

The Inequality Possibility Frontier

Extensions and New Applications

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May 2013



Abstract

This paper extends the Inequality Possibility Frontier approach in two methodological directions. It allows the social minimum to increase with the average income of a society, and it derives all the Inequality Possibility Frontier statistics for two other inequality measures besides the Gini. Finally, it applies the framework to contemporary data, showing that the inequality extraction ratio can be used in the empirical analysis of

post-1960 civil conflict around the world. The duration of conflict and the casualty rate are positively associated with the inequality extraction ratio, that is, with the extent to which elite pushes the actual inequality closer to its maximum level. Inequality, albeit slightly reformulated, is thus shown to play a role in explaining civil conflict.

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The inequality possibility frontier: Extensions and new applications

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Key words: inequality, inequality possibility frontier, civil war

JEL classification: D31, N3, O1, Q3

Number of words: About 7,700

Sector board: Social protection

¹ World Bank, Development Research. Earlier versions of the paper was presented at the conference “Measuring inequality in economic history” held at the Universidad de la Republica in Montevideo in August 2012, economic history seminar at the University of Lund in September 2012, and Poverty and Inequality Measurement and Analysis Practice group seminar at the World Bank, Washington in April 2013. I am grateful to the participants, and in particular to Jorge Alvarez, Luis Bertola, Cecilia Castelnovo, Esteban Nicolini, Javier Rodriguez Weber and Henry Willebald in Montevideo, Ewout Frankema, Christer Gunnarsson and Debin Ma in Lund, and Peter Lanjouw in Washington for very valuable comments. It is written as a part of the KCP-funded research project “Chageable inequalities: facts, perceptions and policies” TF012968. The opinions expressed in the paper are author’s and should not be attributed to the World Bank or its affiliated organizations.

1. Introduction

The purpose of the inequality extraction ratio (Milanovic, Lindert and Williamson 2007, 2011, in further text MLW; first defined in Milanovic, 2006) is to measure how close is measured inequality to the maximum inequality that can exist in a given society (called maximum feasible inequality). The maximum feasible inequality was defined under a special condition such that all but an infinitesimal minority of people (ε) live at the physiological subsistence (s). Then, if we define Y =total income and n =number of people, the surplus (S) over the subsistence will be

$$S = Y - n(1 - \varepsilon)s$$

The surplus will be, under conditions of maximum feasible inequality, received by an infinitesimally small percentage of people, in the extreme case by one person. It should be apparent that the derivation of the maximum feasible inequality follows exactly the same method as used when defining the maximum values of the Gini coefficient or other inequality statistics. The only difference is that here the floor is physiological subsistence (s) rather than income of 0. The conventional maximum Gini of 1 is defined as the situation when all individuals but one have zero incomes, and the one, rich, individual appropriates the entire income of a community. It should also be clear that the conventional maximum Gini is a special case of the maximum feasible Gini when $s=0$.

This can be readily seen from the formal definition of the maximum feasible Gini. As mentioned, the society consists of two groups of people: εn with income $(Y-n(1-\varepsilon)s)/\varepsilon$ and $n(1-\varepsilon)$ with income of s . The Gini coefficient (G^*) is by definition

$$G = \frac{1}{\mu} \sum_{i=1}^n \sum_{j>i}^n (y_j - y_i) p_i p_j$$

where m =mean income, y_i =income of individual or group i , and p_i =share of i 's in total population. With only two groups and average incomes as given, the Gini coefficient reduces to

$$G^* = \frac{1}{\mu} \left(\frac{nm - n(1-\varepsilon)s}{n\varepsilon} - s \right) (\varepsilon)(1 - \varepsilon) \quad (1)$$

Rewriting (1), we get

$$G^* = \frac{1}{m} \left(\frac{1}{\varepsilon} (m - (1 - \varepsilon)s) - s \right) (\varepsilon)(1 - \varepsilon) = \frac{1 - \varepsilon}{m} (m - s) \quad (2)$$

Obviously, when $\varepsilon \rightarrow 0$, the expression simplifies to

$$G^* = \frac{1}{m} (m - s) = 1 - \frac{s}{m} \quad (3)$$

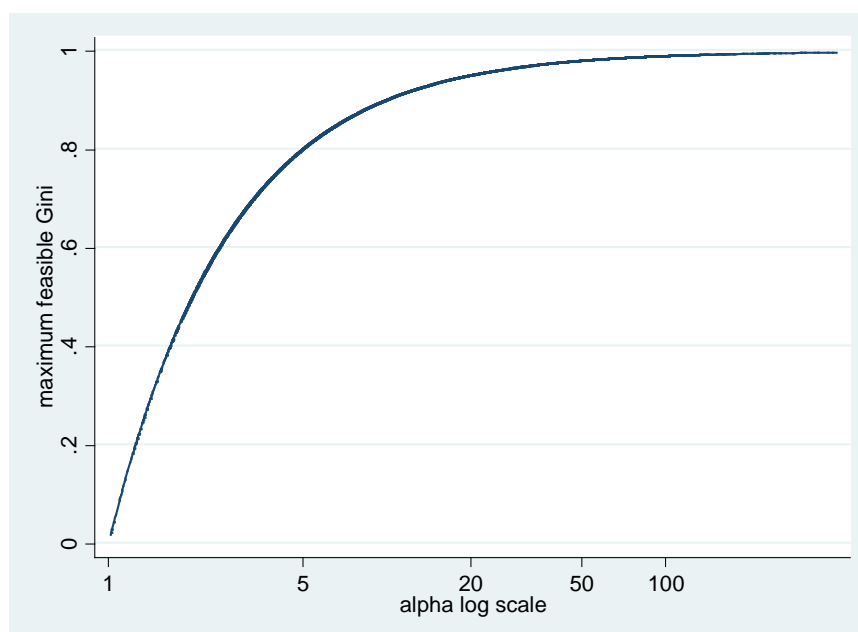
If we express the mean income in terms of the physiological subsistence (which we shall find particularly useful when dealing with pre-industrial economies), and denote this as $\alpha = m/s$, then (3) becomes

$$G^* = 1 - \frac{1}{\alpha} = \frac{\alpha - 1}{\alpha} \quad (4)$$

Equation (4) is our final expression for the maximum Gini under the condition of all but one person receiving the physiological minimum. Obviously, G^* depends on how rich the society is: the richer it is, the more inequality it can theoretically accommodate because the surplus will be greater, and if it is, by assumption, appropriated by one person, inequality will be greater as well. To fix ideas, suppose that $\alpha = 2$; then $G^* = 0.5$; if $\alpha = 3$, then $G^* = 0.66$ etc. For very high values of α , as in today's advanced economies, where $\alpha > 100$, the maximum feasible Gini will approach 1. In other words, in very rich societies, the maximum feasible Gini (G^*) will not differ much from the conventional Gini derived under the condition that $s = 0$.

The G^* 's that are charted as α increases define the locus of maximum Ginis which we call the inequality possibility frontier (IPF). As shown in Figure 1, the Gini inequality possibility frontier is concave. This can also be checked from equation (4) whose second derivative is negative.

Figure 1. Inequality Possibility Frontier



Consider the situation when the minimum is zero: then, the second term in (3) becomes 0, and the maximum Gini is simply the maximum value of the “standard” Gini index, that is 1 *regardless* of the average income of society. Then, the IPF is a straight line fixed at $G=1$ throughout the range of α 's.

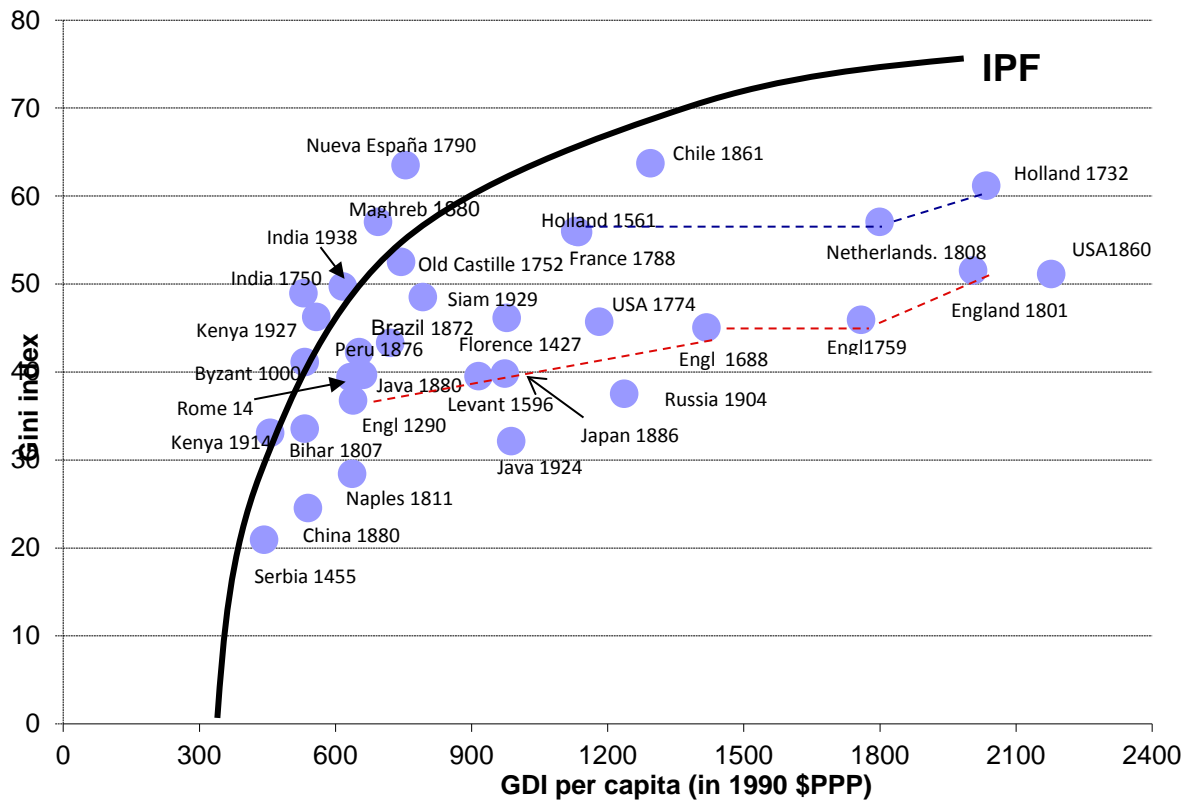
The inequality extraction ratio (IER) is defined as the ratio between the recorded (measured) Gini and the maximum feasible Gini (G^*):

$$IER = \frac{G}{G^*} \quad (5)$$

IER gives an estimate of how close a society is to its inequality possibility frontier. It also implies that the same recorded Gini in two societies that differ in terms of their mean incomes will have very different implications. For a poor society, the measured Gini can be quite close to the maximum feasible Gini, and the IER will be high. For a rich society, whose G^* is much greater, the G/G^* ratio will be lower: the extraction ratio will be less. The ratio will be thus representative both of the level of development of a society and of the ability of the elite to extract the surplus. Thus, to illustrate the role of the average income we may take a contemporary example of Tanzania and Malaysia. Tanzania has a lower Gini than Malaysia (0.38 vs. 0.47 in 2008), but its IER is greater (51 percent vs. 48 percent) simply because Malaysia's income is much higher. And to illustrate the role of predatory elites, we

may take the finding from Milanovic, Lindert and Williamson (2011). The authors find particularly high IERs to have obtained in colonies: inequality there was pushed almost to its maximum, with IERs approaching 100%.² Figure 2 shows the actual Ginis in pre-industrial societies calculated from the social tables drawn against the Inequality Possibility Frontier. The data include new results for the United States in 1774 and 1860 and Russia in 1904 that had become available after the publication of the MLW paper.³ It is easy to notice that the dots representing most of the colonies in the sample (Moghul India 1750 and British India 1938, Nueva España 1790, Maghreb 1880, and Kenya 1912 and 1927) lie around or slightly above the frontier.⁴

Figure 2. Estimated Gini coefficients and the Inequality Possibility Frontier (pre-industrial economies)



So far the derivation of the IER has been done using two important simplifying assumptions. First, it was assumed that the subsistence is an unalterable physiological minimum s . But what would happen if the subsistence itself changes in function of the average income of society, $s=s(\alpha)$, so that it is no longer an absolute minimum but a socially-influenced or socio-cultural minimum? How would our formulas for the maximum Gini and the IER change?

Second, inequality was assumed to be measured by the Gini and the IER was defined as the ratio of the two (actual and maximum feasible) Ginis. The question can be asked: Can the Inequality Possibility Frontier and the IER be derived for other inequality measures and how would they compare with IPF and IER derived for the Gini? Will countries' extraction ratios move the same way whether we measure inequality using one or another measure? In other words, we need to show that the IER is not "a prisoner of the Gini". It is to these two questions, both extensions of the original IER concept, that we turn next.

2. When physiological minimum becomes social minimum

There are three sources of evidence that the social minimum tends to rise with affluence of a community. We use the term "social minimum" to indicate that the "floor income" may rise as mean income increases, while the term "subsistence" or "physiological minimum" is reserved for an "absolute income floor" below which life is not sustainable. To differentiate between the two, we use notation σ for social minimum and s for the physiological minimum.

The first type of empirical evidence comes from the countries' official poverty lines. They may be thought similar to social minimum, and they tend to increase as the mean income of a society goes up (see Chen and Ravallion, 2013, in particular Figure 1, p. 6). Thus, they gradually diverge from the physiological minimum which remains a poverty threshold only in the poorest countries. Chen and Ravallion (2013, p. 9) find that the elasticity of the official poverty lines with respect to mean income is around 0.33, once we move away from the poorest 15 to 20 countries where the elasticity is zero (i.e., their poverty lines are equal

to subsistence).⁵ Converting this cross-sectional regularity into historical terms implies that the physiological minimum was much more likely to have been a “reasonable” minimum in poor, pre-industrial societies than in today’s more affluent ones.

The second type of evidence comes from the studies on subjective poverty. It emerges there that what people consider to be the “minimum income necessary to make ends meet” increases in the respondent’s family per capita income. Most research has yielded the elasticity values between 0.4 and 0.7 (see, e.g. Flik and van Praag, 1991, p. 325; van Praag and Flik, 1992, p. 10). This also accords well with our intuitive perception that as people get richer they set the necessary minimum higher, but do not raise it (in percentage terms) as much as their income increases.

Third, classical economists, most famously Adam Smith, later reprised by Amartya Sen, argued that poverty is not solely a physiological attribute (inability to satisfy some basic minimum needs or functions), but to operate “without shame” in a society. In words of Adam Smith: “By necessaries, I understand, not only the commodities which are indispensably necessary for the support of life, but whatever the custom of the country renders it indecent for creditable people, even of the lowest order, to be without”.⁶ Thus, again, a more affluent society would require a higher social minimum. The same argument was used recently in Ravallion (2012) to argue in favor of a “weakly relative poverty line” for the developing world, such that in addition to the absolute (subsistence) component it would also make allowance for the “social inclusion” needs that rise with the average income of society.

Let the social minimum (σ) be related to the average income normalized by $s (= \alpha)$ as in (6)

$$\sigma = s\alpha^b \tag{6}$$

where b represents the elasticity with which social minimum increases as mean income (normalized by s) of a society goes up. Clearly, if the mean income is at the subsistence itself

⁵ When estimated across all poverty lines and mean consumptions, the estimated elasticity is 0.65 (Chen and Ravallion, 2013, p. 8).

⁶ *The Wealth of Nations*, book 5, ii.

($\mu/s=\alpha=1$), then σ must also be equal to s regardless of the elasticity. As μ increases, σ will diverge from s , depending on the elasticity b . By writing the new expression for σ into (3), we obtain

$$G^*(b, \alpha) = \frac{1}{\mu}(\mu - \sigma) = \frac{1}{\mu}\left(\mu - s\left(\frac{\mu}{s}\right)^b\right) = \frac{1}{\mu}(\mu - s(\alpha)^b) = 1 - \frac{1}{\alpha} \alpha^b \quad (7)$$

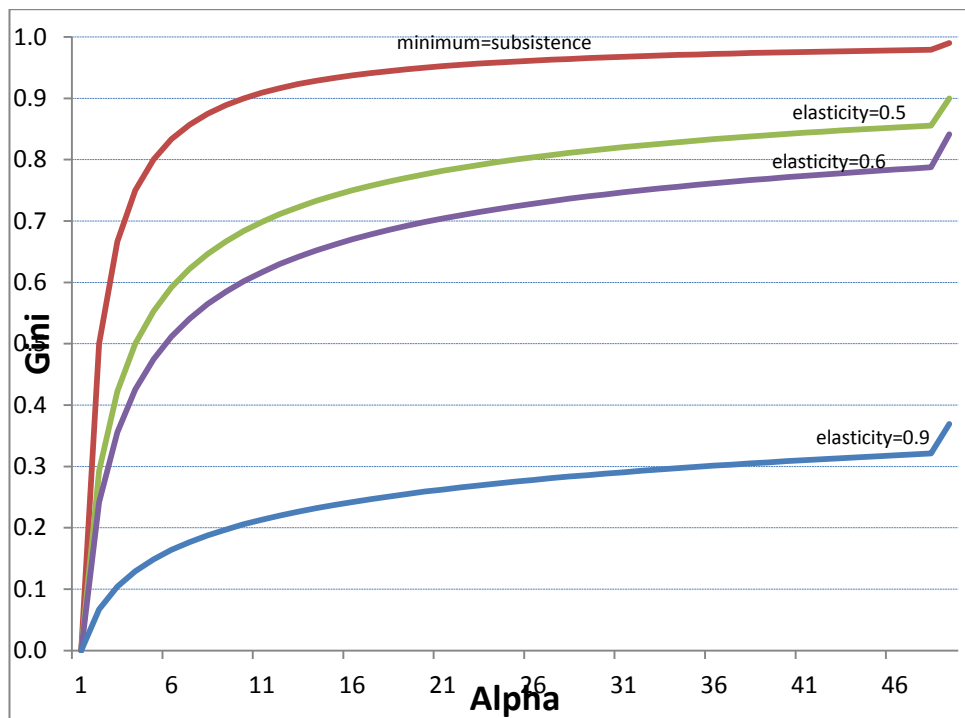
Relationship (7) is the general expression linking the maximum feasible Gini, average income, and elasticity of the minimum with respect to average income. If elasticity $b=0$, the relationship reduces to (3). There is no social element in the minimum and the minimum is purely physiologically determined. At the other extreme, if $b=1$, so that the social minimum increases *pari passu* with the average income, G^* becomes 0. In other words, there cannot be any surplus if it is assumed that all members of a community have to be guaranteed a social minimum equal to the mean income. Then, obviously, everybody has the average income and Gini is equal to zero. Formula (7) is a general case of the maximum feasible Gini which allows subsistence to vary between 0; the physiological minimum s ; or to increase in function of average income growth of a country up to being equal to mean income.

Empirically, b will lie between the two extremes (0 and 1). For simplicity and also based on the subjective poverty literature as well on the observations of how poverty lines rise with the average income (GDP per capita), a reasonable approximation of b may be 0.5.⁷ But higher or lower elasticities are possible. Figure 3 shows the shape of the inequality possibility frontier for several values of b while α ranges from 1 to 48, and then (at the last point on the horizontal axis) attains the value of $\alpha=100$. The curve on the top labeled “minimum= subsistence”, drawn for $b=0$, is the same IPF as the one which we already drew in Figure 1. It is the IPF constructed under the assumption that the subsistence minimum does not include any “socially-influenced” part and is purely physiological. As b increases, and the social minimum begins to (increasingly) respond to the rise in the average income, the IPF shifts downward. This should be easy to understand intuitively. As we require that all members of a community have at least an income that rises in proportion to the mean, the “surplus” to be divided among an infinitesimally small elite, will be less. That surplus will

⁷ Notice also that in that case, equation (7) for the maximum feasible Gini conveniently simplifies to $1 - \frac{1}{\sqrt{\alpha}}$. For example, if $\alpha=100$, as in today’s rich economies, $G^*=0.9$.

obviously be less the greater the elasticity of the social minimum with respect to the mean. In consequence, maximum feasible inequality must be less and its locus, IPF must shift downward (for any given α).⁸

Figure 3. Inequality possibility frontier for different values of the social minimum



The introduction of a social, rather than a merely physiological minimum, which in turn shifts downward the IPF, has a straightforward implication for the inequality extraction ratio. Since it is the ratio between measured and maximum feasible Gini, and the latter is now lower, the IER increases.

Table 1 and Figure 4 show historical IER for England and Wales (or United Kingdom) and the United States under two assumptions: (a) that the subsistence minimum is constant and (b) that it increases in proportion to the mean income with elasticity of 0.5. Figure 4 shows the inequality extraction ratios over a long historical period covering more than 300 years for England/UK and 200 years for the United States. The usefulness of the IER ratio appears most obviously in such examples because of the big variations in real incomes over such long periods. For the UK, real per capita income between 1688 (the first year for which

⁸ The concave shape of IPF (for any given b) remains.

we have inequality estimates) and 2010 increased by more than 16 times.⁹ The figure shows the large difference between the IERs calculated with subsistence only, and under the assumption that the social minimum increases as the mean income goes up. Under the first assumption, the IER in England/UK oscillated around 60 percent throughout 18th and 19th century. But if we assume that the social minimum had risen in proportion with real income ($b=0.5$), the IERs attained 80 percent. The numbers for the United States are very similar.

The political implications of $b=0$ and $b=0.5$ are very different. If the elites had to acquiesce to a rise of the minimum living standard for the masses, then an 80% inequality extraction ratio shows that they were able or willing to push overall inequality rather close to the maximum. If, on the other hand, we believe that there was no social pressure to increase the social minimum, then the elites' appetites may be thought to have been relatively modest.

In both countries, it is only in the 20th century, as illustrated more clearly in Figure 5, that the IER, calculated under the more stringent conditions of a changing social minimum, began its downward slide, to be arrested and reversed in the last quarter of the 20th century and early 21st. But because real incomes have by then risen to very high levels (exceeding, in the United States, 100 times the subsistence minimum), the inequality possibility frontier was close to 1 whether we use $b=0.5$ or $b=0$ assumption, and the inequality extraction ratio was similar under both scenarios.¹⁰ Today, actual inequality in the UK reaches the level of some 40 percent of the maximum feasible inequality while in the United States it is just over 50 percent. This is a significant improvement compared to the situation some 150 years ago when the IER ratios in both countries were around 80 percent.

While the changes in the IER are more dramatic and probably more informative when calculated over longer periods of time, the results can be very interesting for the contemporary period as well particularly when we are dealing with countries that register

⁹ GDP per capita data, expressed in 1990 Geary-Khamis PPPs, are from Maddison (2007). The physiological subsistence minimum in the same prices is assumed to be \$PPP 300 (for discussion see MLW 2011, p. 262).

¹⁰ The maximum feasible Gini is bounded from above (at 1), and the gap between the maximum feasible Ginis calculated for different b 's is decreasing in α , as can be seen in Figure 3. To see this, define, using expression (3), the "gap function" between the two maximum Ginis for two different b 's as, $G^*(b_1, \alpha) - G^*(b_0, \alpha)$ where $b_1 > b_0$ and differentiate it with respect to α . It can be easily verified that the "gap function" is decreasing in α .

high income increases or declines. Recently, perhaps the most startling is the comparison between China and Russia.

Figure 6 shows their diverging patterns in the 1990s and in the first decade of the 21st century (assuming in both cases that the subsistence is fixed at the absolute level). In both Russia and China, as shown in the left-hand panel, Gini increased substantially. In the case of China this happened against the background of an almost seven-fold increase in GDP per capita while in Russia it happened while GDP per capita at first went steeply down and then recovered, reaching in 2009 the same level as 21 years earlier. This had very different implication for the inequality extraction ratio (see right-hand side panel). In China, higher GDP per capita offset higher inequality and the IER remained at 50 percent. In Russia, the extraction ratio almost doubled going up from 21 percent to 41 percent. These patterns would be somewhat less striking if we assumed that the social minimum changes with mean income: both Chinese gains and Russian loses would be less dramatic.

Table 1. Historical inequality extraction ratios
for England/United Kingdom and the United States, selected years

England/United Kingdom

Year	Estimated Gini	Estimated α (GDP per capita/s)	Maximum Gini		Inequality extraction ratio (in percent)	
			With s only	with b=0.5	With s only	with b=0.5
1209	36.7	2.1	53.1	31.5	69	117
1688	45.0	4.7	78.8	54.0	57	83
1759	45.9	5.9	82.9	58.7	55	78
1801	51.5	6.7	85.0	61.3	61	84
1867	53.0	9.9	89.9	68.2	59	78
1964	33.6	31.9	96.9	82.2	35	41
1979	28.8	43.9	97.7	84.9	29	34
1991	35.9	53.9	98.1	86.4	37	42
2002	37.4	69.8	98.6	88.0	38	42
2007	37.1	78.8	98.7	88.7	38	42
2010	37.4	76.2	98.7	88.5	38	42

United States

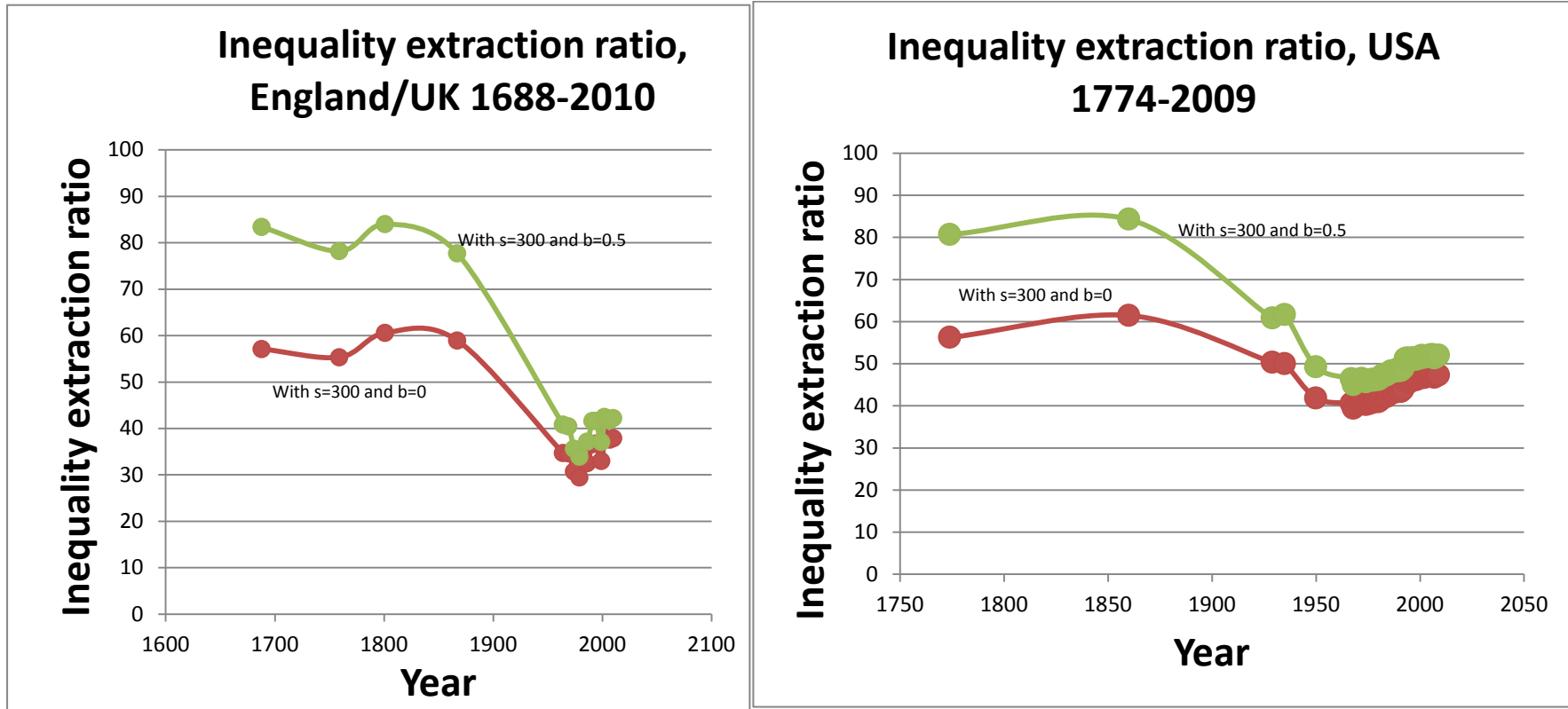
Country/year	Estimated Gini	Estimated α (GDP per capita/s)	Maximum Gini		Inequality extraction ratio (in percent)	
			With s only	with b=0.5	With s only	with b=0.5
1774	45.6	5.3	81.1	56.5	56	81
1860	53.0	7.3	86.2	62.9	62	84
1929	48.1	23.0	95.7	79.1	50	61
1935	47.2	18.2	94.5	76.6	50	62
1950	40.5	31.9	96.9	82.3	42	49
1967	39.7	47.8	98.0	85.5	41	46
1979	40.4	62.6	98.4	87.4	41	46
1991	42.8	76.2	98.7	88.5	43	48
2002	46.2	95.3	99.0	89.8	47	51
2007	46.3	104.5	99.0	90.2	47	51
2009	46.8	99.4	99.0	89.7	47	52

Sources: GDP per capita for both countries from Maddison (2007), expressed in 1990 PPPs. The subsistence minimum assumed \$PPP 300 in 1990 international prices.

UK Gini: 1209-1801 from Milanovic, Lindert and Williamson (2011) based respectively on social tables drawn by Campbell (2007) for year 1209, Gregory King for 1688 [1696], Joseph Massie for 1759 [1760] and Patrick Colquhoun for 1801-3 [1806]. Data for 1867 calculated from Lindert and Williamson (1983) which are in turn based on social tables produced by Dudley Baxter [1869]. UK Gini from 1967 onward from Luxembourg Income Study (income concept: disposable household per capita income across individuals).

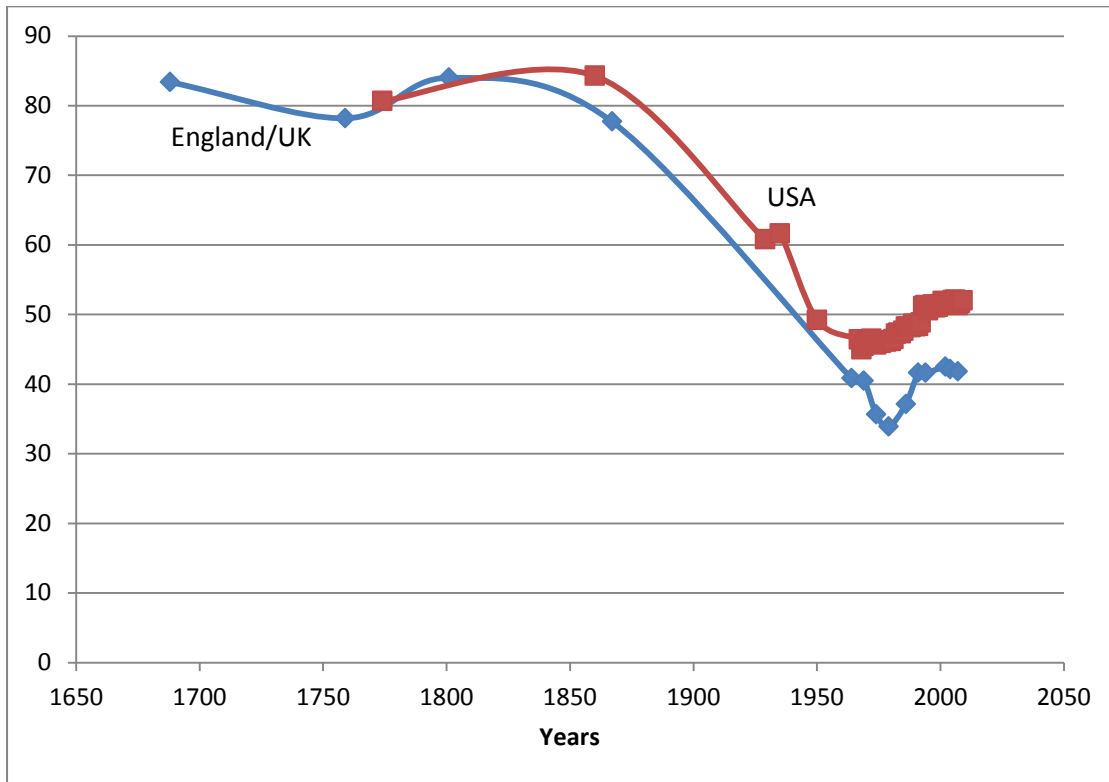
US Gini: 1774 and 1860 from Lindert and Williamson (2011). Period 1929-50 from Goldsmith et al (1954, fn. 4, p. 7). Gini after 1967 from US Census Bureau (2010). The US concept is gross household income across households.

Figure 4. Inequality extraction ratios with $b=0$ and $b=0.5$



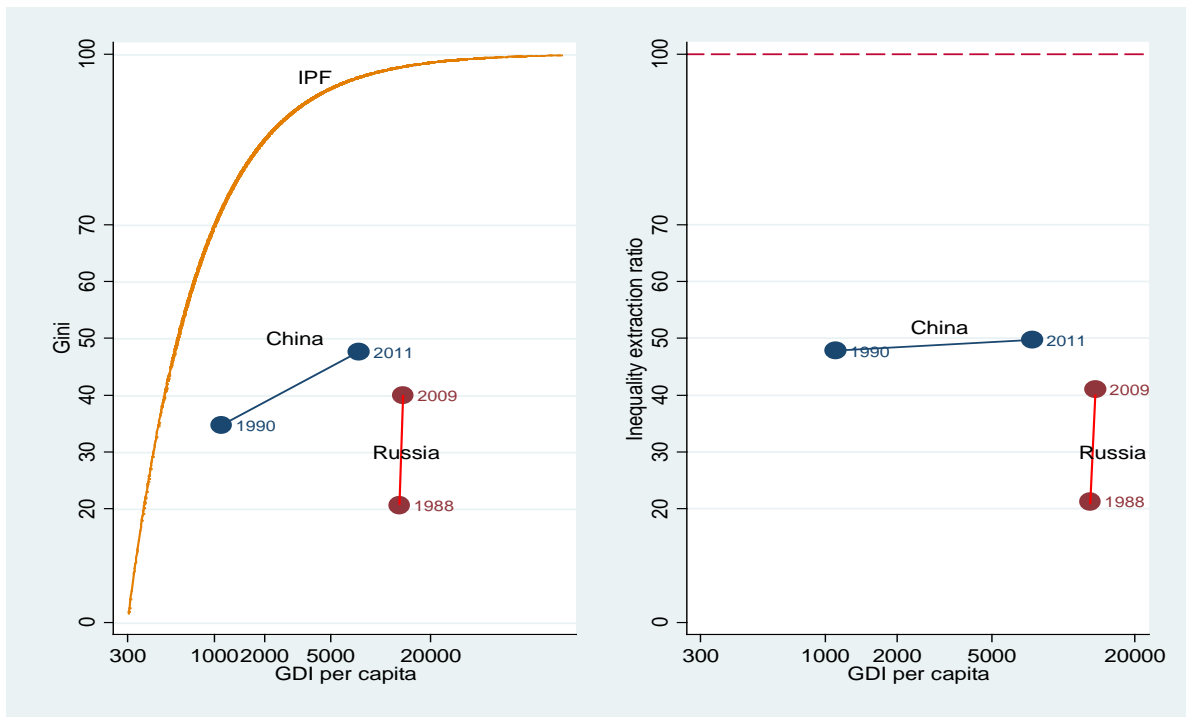
Source: See notes to Table 1.

Figure 5. UK and US historical inequality extraction ratios
(elasticity of the social minimum with respect to mean income = 0.5)



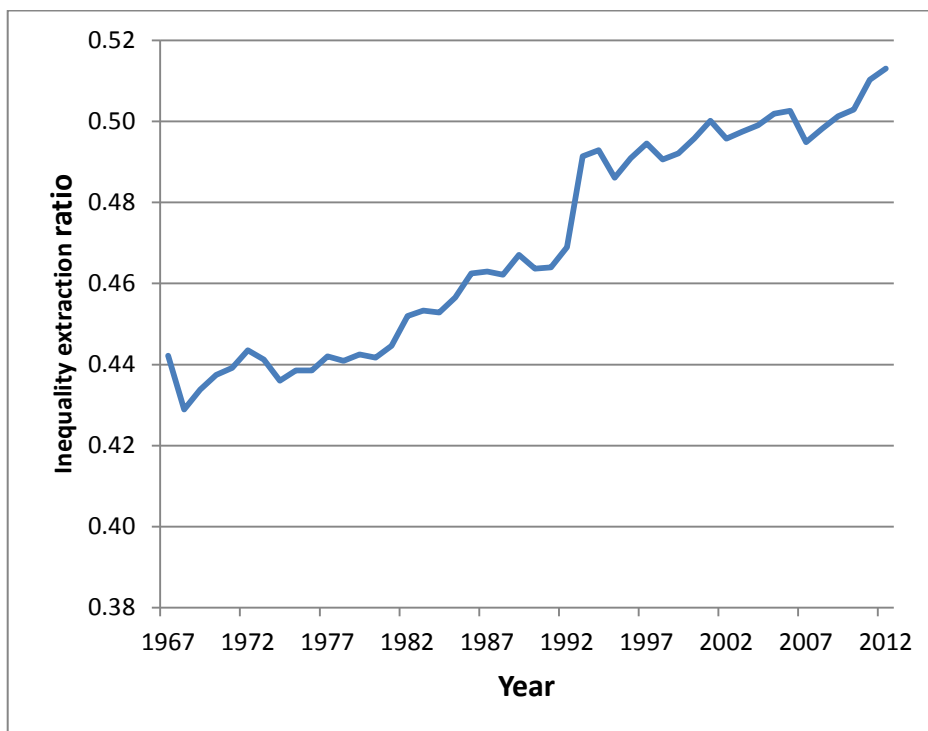
Source: See notes to Table 1.

Figure 6. Inequality possibility frontier (left) and the inequality extraction ratios (right panel) for China and Russia



Another way to look at the behavior of the inequality extraction ratio with a changing social minimum is to take the current actual social minima, find out what they implicitly imply about b , and calculate the inequality extraction ratio based on such “real” data (that is, without a prior assumption of a given elasticity). Take the United States in 2012. Its GDP per capita was \$43,000 and the federal poverty line for a four-member household was \$23,050. On a per capita basis, this gives a poverty line of \$5,762 per year. Assuming as before a physiological subsistence of \$300 per year, and plugging these actual values into (6), enables us to calculate the implicit elasticity b . It works out as 0.59. Figure 7 shows the inequality extraction ratio for the United States over the period 1965-2012 using this implicit observed elasticity of the social minimum with respect to real income. The pattern of the increase is the same as with $b=0.5$, but the level of the extraction ratio is higher. In 2012, it is about 3 points higher: 51 percent instead of 48 percent. Of course, in either case, the rise of the extraction ratio during the past 45 years was substantial.

Figure 7. Inequality extraction ratio in the United States, 1967-2012 (calculated using the observed elasticity between the social minimum and GDP per capita, $b=0.59$)



3. The Inequality Possibility Frontier with measures other than Gini

So far the analysis has been conducted exclusively in terms of the Gini coefficient. It is around the Gini coefficient that we have created the inequality possibility frontier and the inequality extraction ratio. In other words, we had assumed throughout that inequality = Gini. But that of course is not true. Inequality can be measured by many other measures. The question can then be asked: Would our results remain if instead of Gini we used another measure of inequality? That implies re-expressing the entire framework (i.e., deriving IPF and IER) in terms of other inequality measures. We do this here for three measures: the two Theil indices, and the standard deviation of logs of incomes.

The first step is the derivation of the maxima for each of the measures and for different α 's, under the assumption that the subsistence is fixed in absolute amounts. Annex 1 shows the derivations of the inequality possibility frontier for Theil (0), Theil (1) or Theil's entropy measure, and the standard deviation of log of incomes.¹¹ Table 2 shows the formulas, the range of each measure, and the final expression for the inequality possibility frontier. For both Theil (0) and the standard deviation of logs, the maximum feasible inequality is equal to α , that is, is directly proportional to the mean income (expressed in multiples of subsistence). For Theil (1), there is no upper bound to the maximum and hence the Inequality Possibility Frontier cannot be defined.¹²

Figure 8 shows the Inequality Possibility Frontiers and the calculated Gini and Theil (0) coefficients for the same sample of pre-industrial economies. Two things stand out. First, results with Theil generally show colonies (full dots) at a greater distance from the Inequality Possibility Frontier than when we use Gini. The inequality extraction ratio, calculated across nine colonies in the sample is 90 percent with Gini and 55 percent with Theil. In other words, within the Theil framework, it would seem that the elites in colonies, while exploitative, have not nearly exhausted the entire surplus as implied by the Gini framework. Part of the reason may lie in the difference in sensitivities to various parts of

¹¹ Note that this measure of inequality (standard deviation of logs of incomes) is also a key functional parameter if incomes are lognormally distributed.

¹² This is an interesting reversal: the range of Theil (0) coefficient is $[0, \infty)$ but the maximum Theil (0) and hence the IPF are well defined. On the other hand, the range of Theil (1) is $[0, \ln n]$ where n =number of observations, but the maximum Theil (1) is not bounded from above and hence the IPF cannot be defined.

the income distribution exhibited by the two measures. As is well-known, Theil is much more sensitive to extreme values, while Gini is most sensitive to the values around the mode. Since our data for pre-industrial economies are calculated from the social tables with a limited number of social classes, the top of the distribution is truncated, or more exactly “squeezed”. The top is represented by the mean income of the richest class, not by the individual incomes of the richest individuals. Hence, Theil may tend to show much lower values than if we had more finely-grained (individual-level) data. The gap between the measured and maximum Theil may thus be overestimated.

Second, for high values of α the distance of the calculated (actual) Theils from the Inequality Possibility Frontier is much greater than the analogous distance of the Gini. The reason is that the IPF within the Theil framework increases linearly in logs while within the Gini framework IPF is concave. Thus, for example, for countries with $\alpha > 4$, the average IER using Gini is 66 percent; using Theil, it is less than half (30 percent).¹³ But while the levels of IER calculated with Gini and with Theil differ, the correlation between the two IER measures is quite high: both the linear and rank correlations are 0.89.

In conclusion, Gini and Theil seem to rank countries according to their inequality extraction ratios very much alike: more “exploitative” countries will be deemed such by both approaches. But the implied levels of IER are significantly lower with Theil than with Gini. This, in turn, has implications on our view regarding how close to the inequality frontier were different societies. It also seems that, given the nature of social tables that are used to estimate pre-industrial inequalities, Gini would be more likely to capture inequality well, and in this case at least, it would be reasonable to prefer it to Theil.

¹³ There are 7 countries with $\alpha > 4$.

Another way to check whether IERs calculated from Gini and Theil (0) behave similarly is to use the regression results from Milanovic, Lindert and Williamson (2007) derived within Gini-based framework and check if they “survive” when we use IERs based on Theil. MLW (2007) find, on the sample of 28 pre-industrial societies, that two variables are strongly associated with the level of inequality extraction: being a colony increases the IER by some 25 Gini points; countries that are more densely populated are associated with lower IER (an increase of 10 people per km² reduces IER by 1.9 points).¹⁴ As shown in Table 3, both variables remain highly statistically significant when the IER is calculated within the Theil framework.¹⁵ The absolute values of the coefficients are slightly different because the values of Theil indexes and Theil-based IERs differ from those of Gini and Gini-based IERs. But both approaches yield strikingly similar results. In other words, and this is very important, IER results (in this case at least) do not depend on whether we situate ourselves within the Gini or Theil framework.

Table 3. Explaining the Inequality Extraction Ratio:
dependent variable = measured inequality (Gini or Theil) / maximum feasible inequality (Gini or Theil)

	Gini-framework	Theil-framework
GDP per capita (in ln)	128.22 (0.34)	151.26 (0.49)
Squared (ln) GDP per capita	-9.81 (0.32)	-11.46 (0.48)
Percent of urban population	0.31 (0.24)	0.21 (0.65)
Population density per km ²	-0.19** (0.000)	-0.20** (0.001)
Colony dummy	26.61** (0.000)	24.18** (0.002)
Constant	-344.46** (0.46)	-455.26 (0.54)
Adjusted R ² (F)	0.67 (12.1)	0.53 (6.6)
No. of observations	28	28

Note: ** Indicates statistical significance at less than 1%. *p*-values between brackets.

¹⁴ The finding regarding the role of population density is probably the most interesting because it lends itself to several interpretations (see MLW, 2007). The effect of colonies is rather expected.

¹⁵ In both cases, IERs are calculated under the assumption that the subsistence minimum is fixed in real terms at \$PPP 300 (in 1990 international dollars).

4. A contemporary application: Explaining conflict

One of the ways to look at the contemporary relevance of the inequality extraction ratio is to study its potential role in one of the areas where, on an *a priori* grounds, it should matter, namely in the analysis of civil war and within-national conflict. As is well known, simple interpersonal inequality reflected in a Gini coefficient is seldom found to be a statistically significant determinant of conflict (see Collier and Hoeffler 2004; Collier, Hoeffler and Rohner 2008, p. 16; and review in Sambanis 2004) . This leads to a somewhat bizarre and counter-intuitive conclusion that inequality is not associated with conflict. More recently, other types of inequalities, most notably horizontal, that is, inequality in average incomes between groups, have gained prominence (Stewart 2000; Østby 2008; Cederman, Weidmann and Gleditsch, 2011). While in principle such inequality may be thought related to conflicts in multi-ethnic or multi-religious societies, it cannot capture inequality in societies that are homogeneous along these dimensions. For example, the civil war waged by the Khmer Rouge in Cambodia was dominantly politically-driven, not ethnic. So were the Great Leap Forward, the first civil war in Angola between the government and UNITA and a number of conflicts in Argentina and Colombia.

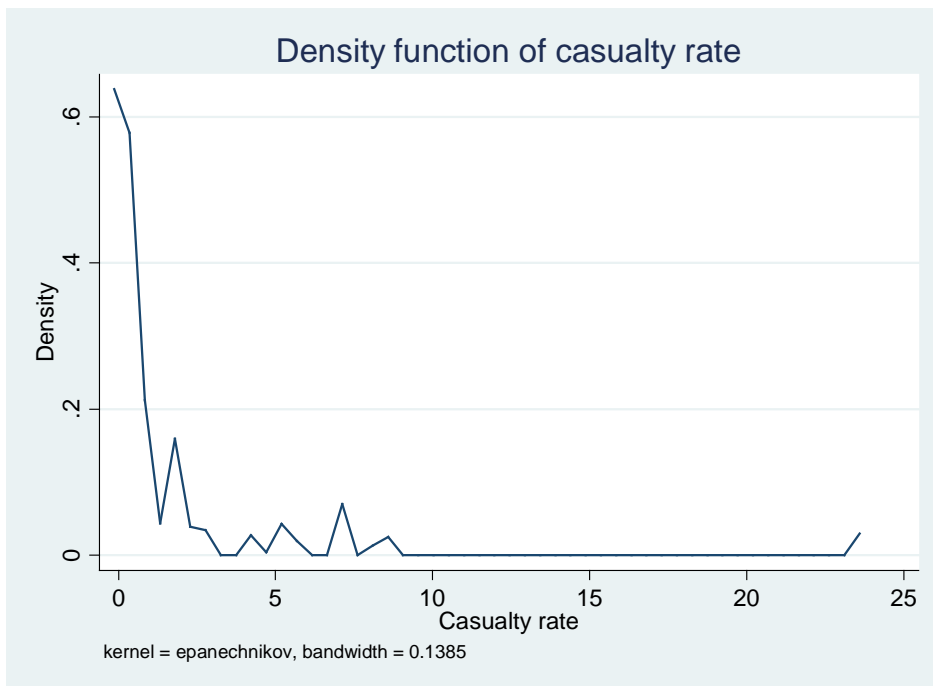
The inequality extraction ratio, by capturing how close to the frontier is actual inequality, conveys the information about the relative “rapaciousness” of the elite and combines in its formulation two aspects that are often found important for the explanation of civil conflict: the average level of development of a country (its GDP per capita) and its income distribution. To check the role of IER, we use a very detailed database on civil war created by Nicholas Sambanis (Sambanis and Schulhofer-Wohl, 2009). The database covers 151 conflicts in 70 countries over the period 1945-2002. This is probably the most complete data base of civil conflict: it treats as distinct the conflicts that might have overlapped in a given country over the same time period but were motivated by different reasons and had different actors; it includes the data on the duration of each conflict and its estimated casualties. Here we consider the role of IER in “explaining” the number of years of conflict per country and the overall casualty rate in the period 1960-2002. We exclude the pre-1960s conflicts that were often driven by the decolonization movement and whose determinants differed from those of the post-1960s conflicts.

The results are shown in Table 4. The regression is run across countries with the dependent variable summing or averaging the conflict outcomes for the period 1960-2002. The years of war variable shows the total number of years of civil war (with a year coded as being a “civil war year” even if conflict lasted only one month). The variable has, unsurprisingly, a mode at 0 with 116 countries (out of the sample of 185 countries) not experiencing civil conflict. The maximum value is 41 years for the Philippines and Colombia. The other dependent variable, the casualty rate, is the ratio between total estimated casualties from civil war(s) divided by the average population size during the 1960-2002 period. The casualty rate is obviously 0 for countries without civil wars, and its maximum value is 23 percent in Cambodia followed by 8.4 percent in Angola and 8 percent in Afghanistan. Even when we exclude countries with a zero casualty rate, the density function is heavily skewed to the right and is strongly “bunched” around very small (less than 1 percent) values (see Figure 9).

The distribution of the inequality extraction ratio in countries with no civil conflict and those with at least one conflict is shown in Figure 10. The latter distribution is shifted to the right, with a higher mean and median. In countries without conflict, the mean extraction ratio is 41.3 percent and the median 36.9 percent; in countries with civil conflict, the mean is 50.6 percent and the median 48.1 percent.¹⁶ Equality of the means and medians is rejected at less than 1% level. So is the equality of the two distributions by the Smirnov-Kolmogorov test.

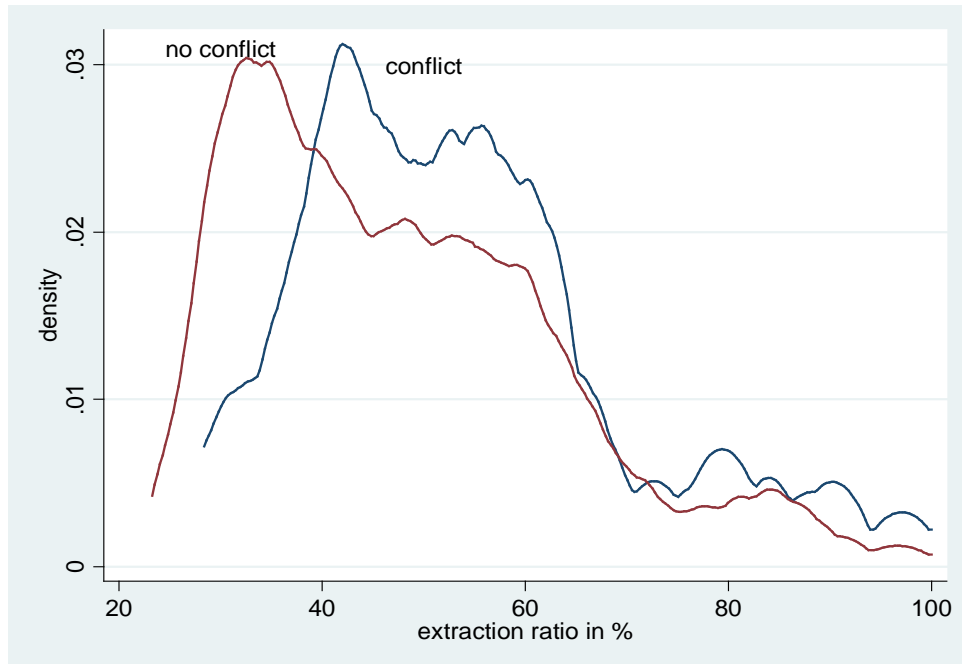
¹⁶ The standard deviations however are very close: 14.4 and 14.8 percent. χ^2 test accepts their equality.

Figure 9. Distribution of the casualty rate from civil conflict over the period 1960-2002



Note: Casualty rate = total number of dead over 1960-2002 divided by the average population over the same period (in percent).

Figure 10. Distribution of the inequality extraction ratio in countries with no civil conflict and in countries with a positive number of civil conflicts



As shown in Table 4, the introduction of IER to replace both GDP per capita and Gini does not reduce the explanatory power of the models.¹⁷ The inequality extraction ratio is positively related to the civil war variables, and for the number of years of civil war, it is statistically highly significant. If we compare IER with Gini only, in both instances, IER outperforms Gini. While in the explanation of the casualty rate, the IER is not statistically significant, its sign is positive while Gini is marginally negative. For the duration of conflict, the difference is small: both are significant, but IER is significant at a lower p -level. In interpreting the role of IER, we conclude that a ten percentage point increase in IER is associated with, on average, an additional 0.2 years (2.4 months) of civil war, and with 0.12 percent increase in the overall casualty rate. Finally, we note that, as expected, ethno-linguistic fractionalization is strongly correlated with both duration of civil wars and casualty rates, while the effect of democracy is rather ambiguous: it is negatively associated with casualty rates but positively with years of conflict.

Table 4. Regressing civil war related variables

	Number of years of civil war		Overall casualty rate (ln)	
GDP per capita (ln)	-0.665** (0.000)		-0.322 (0.073)	
Gini (in %)	0.010* (0.013)		-0.0001 (0.994)	
Inequality extraction ratio (in %)		0.020** (0.000)		0.012 (0.192)
Democracy (Polity measure)	0.148** (0.000)	0.027* (0.036)	-0.772 (0.176)	-0.133** (0.002)
Ethno-linguistic fractionalization	3.661** (0.000)	3.398** (0.000)	8.193** (0.000)	8.262** (0.000)
Constant	5.595** (0.000)	0.180 (0.228)	-0.794 (0.633)	-3.822** (0.000)
R ² or pseudo R ²	0.32	0.25	0.38	0.38
Number of observations	143	143	143	143

Note: For number of years of civil war we use Poisson regressions. When transforming the casualty rate into logs, 0 casualty rate is treated as 0.01. IER is expressed in percentage points (G/G* times 100); so is Gini. Democracy is measured by the democracy variable from PolityIV database. It ranges from 0 (absence of democracy) to 10 (full democracy). All explanatory variables are 1960-2002 country averages. ** (*) Indicates statistical significance at less than 1% (5%). p -values between brackets.

¹⁷ Both Gini and the IER are expressed in percentages which facilitates the comparison between the two.

5. Summary and conclusions

The paper had two key objectives: to extend the methodological work done on the inequality possibility frontier by allowing for a flexible social minimum that changes with the increase in the average income of a society, and to derive the inequality possibility frontier and the inequality extraction ratio for other inequality measures than Gini.

The inclusion of an income-flexible social minimum makes the surplus, at any average income level, less than if the subsistence is entirely physiologically determined. The IPF thus shifts downward, and the IER becomes greater. The difference between the two IERs is particularly marked in the past when the average incomes were much lower. Thus, over the 18th and 19th centuries, the inequality extraction ratios in the UK and the United States, calculated using a social minimum that increases modestly with the average income ($b=0.5$), in percent, were in the 80's. They were much lower, in the 60's, when calculated with a simple physiological minimum. The political implication of the finding is that, once the elite had to concede an increase in living standards of the poor as the economy got richer, it was quite efficient or ruthless in maximizing inequality to a very high degree. This is however different from the present-day situation. Using the observed elasticity of the social minimum with respect to the mean income in the United States of 0.59, the IER, despite its recent increase, turns out to be just above 50 percent.

The use of other inequality measures, like Theil, in the IPF framework is both feasible (although Theil's entropy measure cannot be used since its maximum is not bounded from above) and consistent with the results obtained within the Gini framework. We thus find that using either Theil (0) or Gini does not make any difference in our conclusion that, among pre-industrial economies, colonies were significantly more "exploitative" while densely populated countries were significantly less. This finding has political implications too. If more densely populated countries were less unequal, was it because they somehow escaped the Malthusian trap, and then the populace by its sheer multitude presented enough of a threat to those in power to deter them from a more exploitative behavior? It is a question that needs further research, perhaps leading to results that may prove important for explaining countries' historical inequality and growth trajectories.

The usefulness of the IPF approach was also illustrated on the contemporary example of factors associated with civil conflict. We find that IER can successfully replace GDP per capita and Gini in such explanations without reducing the explanatory power of the models. In other words, instead of an elusive role of the Gini coefficient in explaining civil conflict, we argue that the extraction ratio brings inequality (albeit formulated somewhat differently) back to the center stage.

For the IPF framework to prove valuable, further work will have to assess its empirical relevance. We need to know much more about whether the inequality extraction ratio can be usefully deployed to answer historical questions and to address the role of inequality today, particularly in poorer societies, where the Inequality Possibility Frontier is more binding.

REFERENCES

- Bruce Campbell (2007), "Benchmarking Medieval Economic Development: England, Wales, Scotland and Ireland c. 1209", *Economic History Review*, 2007, p. 1-50; Table 17, p. 45.
- Lars-Erik Cederman, Nils Weidmann and Kristian-Skrede Gleditsch (2011), "Horizontal Inequalities and Ethnonationalist Civil War: A Global Comparison", *American Political Science Review* 105 (3), pp. 478-495.
- Shaohua Chen and Martin Ravallion (2013), "More Relatively-poor People in a Less Absolutely-poor World", Fifth Ruggles Lecture for the International Association for Research in Income and Wealth, *Review of Income and Wealth*, March 2013, pp. 1-28.
- Paul Collier and Anke Hoeffler (2004) "Greed and Grievance in Civil War", *Oxford Economic Papers*. 56(4): 563-595, p. 577.
- Paul Collier, Anke Hoeffler and Dominic Rohner (2006), "Beyond Greed and Grievance: Feasibility and Civil war", CSAE Oxford, Working Paper 2006/10, August. Available at <http://www.csae.ox.ac.uk/workingpapers/pdfs/2006-10text.pdf> (accessed February 16, 2013).
- Fearon, J.D. and Laitin, D.D. (2003). "Ethnicity, Insurgency, and Civil War", *American Political Science Review* 97(1): 75-90; Collier, Hoeffler and Rohner (2008)
- Robert Flik and Bernard van Praag (1991), "Subjective Poverty Line Definitions", *De Economist*, vol. 139, No. 3, 1991, pp. 311-331.
- Selma Goldsmith, George Jaszi, Hyman Kaitz and Maurice Liebenberg (1954), "Size Distribution of Income since the Mid-thirties", *Review of Economics and Statistics*, vol. XXXVI, No. 1, February.
- Peter Lindert and Jeffrey Williamson (1983), "Reinterpreting Britain's Social Tables, 1688-1913", *Explorations in Economic History*, vol. 20, pp. 94-109.
- Peter Lindert and Jeffrey Williamson (2011), "American Incomes Before and After the Revolution", National Bureau of Economic Research Working Paper No. 17211. Available at <http://gpih.ucdavis.edu/files/w17211.pdf> (accessed February 5, 2013).
- Angus Maddison (2007), *Contours of the World Economy, 1-2030: Essays in Macro-economic History*, Oxford: Oxford University Press. Data available at <http://www.ggdc.net/maddison/oriindex.htm> (accessed February 16, 2013).
- Branko Milanovic, Peter Lindert and Jeffrey Williamson (2011), "Pre-industrial Inequality", *Economic Journal*, March 2011, pp. 255-272.
- Branko Milanovic (2006), "An Estimate of Average Income and Inequality in Byzantium Around Year 1000", *Review of Income and Wealth*, No. 3, September, pp. 449-470.

Branko Milanovic, Peter Lindert and Jeffrey Williamson (2007), "Measuring Ancient Inequality", World Bank Policy Research Working Paper No. 4412, November 2007.

Steven Nafziger and Peter Lindert (2012), "Russian Inequality on the Eve of Revolution", National Bureau of Economic Research Working Paper No. 18383, September. Available at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2147094 (accessed February 5, 2013).

Bernard van Praag and Robert J. Flik (1992), "Subjective Poverty", Foundation for Economic Research, Rotterdam, Research Institute for Population Economics. Final report in the framework of European Eurostat Project "Enhancement of family budget surveys to derive statistical data on least privileged groups."

Gudrun Østby (2008), "Polarization, Horizontal Inequalities and Civil Conflict", *Journal of Peace Research* 45(2), pp. 143-162.

Martin Ravallion (2012), "Benchmarking Global Poverty Reduction", World Bank Policy Research Working Paper No. 4703, September.

Nicholas Sambanis, "Poverty and the Organization of Political Violence: A Review and Some Conjectures." In Susan M. Collins and Carol Graham, eds., *Brookings Trade Forum 2004*. Washington, DC: Brookings Institution, 2005, pp. 165-222.

Nicholas Sambanis and Jonah Schulhofer-Wohl (2009). "What's In A Line? Is Partition the Solution to Civil War?". *International Security*, 34 (2): 82-118.

Frances Stewart (2000), "Crisis Prevention: Tackling Horizontal Inequalities", *Oxford Development Studies* 28(3), pp. 245-62.

US Census Bureau (2010) "Income, Poverty and Health Insurance Coverage in the United States:2009", September 2010.

Annex 1. Derivation of the Inequality Possibility Frontier for three other measures of inequality

A. Derivation of the maximum Theil (Theil 0, or mean log deviation) for a given α

$$(1) T^*(0) = \frac{1}{n} \sum_i \ln \frac{1/n}{y_i/mn}$$

Where n = total population, m = mean income, y_i = income of i -th individual. There are two groups of people: $n(1-\varepsilon)$ people with $y=s$, another group of ε people with income y_h

$$(2) y_h = \frac{nm - n(1-\varepsilon)s}{\varepsilon n} = \frac{nm - ns + n\varepsilon s}{\varepsilon n} = \frac{m - s + \varepsilon s}{\varepsilon}$$

Substitute (2) into (1)

$$T^* = \frac{n(1-\varepsilon)}{n} \ln \frac{1/n}{s/mn} + \frac{\varepsilon n}{n} \ln \frac{1/n}{\frac{m-s+\varepsilon s}{\varepsilon}} = (1-\varepsilon) \ln \frac{mn}{sn} + \varepsilon \ln \frac{mn\varepsilon}{n(m-s+\varepsilon s)}$$

$$T^* = (1-\varepsilon) \ln \alpha + \varepsilon \ln \frac{m\varepsilon}{m-s+\varepsilon s} = (1-\varepsilon) \ln \alpha + \varepsilon \ln \frac{\alpha s \varepsilon}{\alpha s - s + \varepsilon s} = (1-\varepsilon) \ln \alpha + \varepsilon \ln \frac{\alpha \varepsilon}{\alpha - 1 + \varepsilon}$$

$$\lim_{\varepsilon \rightarrow 0} T^* = \ln \alpha + \varepsilon \ln \frac{\alpha \varepsilon}{\alpha - 1 + \varepsilon}$$

The latter expression tends to $0 \times -\infty$. Transform so that it can be solved by L'Hôpital's rule

$$(3) \frac{\ln \frac{\alpha \varepsilon}{\alpha - 1 + \varepsilon}}{\frac{1}{\varepsilon}} = \frac{\left\{ \frac{\alpha}{\alpha - 1 + \varepsilon} - \frac{\alpha \varepsilon}{[\alpha - 1 + \varepsilon]^2} \right\} \frac{\alpha - 1 + \varepsilon}{\alpha \varepsilon}}{\frac{-1}{\varepsilon^2}} = \frac{\frac{\alpha A - \alpha \varepsilon}{A^2} \frac{A}{\alpha \varepsilon}}{\frac{-1}{\varepsilon^2}} = \frac{-(A - \varepsilon)\varepsilon}{A}$$

where $A = \alpha - 1 + \varepsilon$

When $\varepsilon \rightarrow 0$, $A \rightarrow \alpha - 1$.

When $\varepsilon \rightarrow 0$, then the whole expression tends to $\frac{-(\alpha - 1)\varepsilon}{(\alpha - 1)} = -\varepsilon = 0$

Consequently, the maximum Theil (0) for a given alpha is $T^* = \ln(\alpha)$.

Note that when $\alpha=1$, the maximum $T^*=0$.

B. Derivation of the maximum Theil (Theil 1, or Theil entropy index) for a given α

$$(1) T^*(1) = \frac{1}{n} \sum_i \frac{y_i}{m} \ln \frac{y_i}{m}$$

$n(1-\varepsilon)$ people with $y=s$.

$n\varepsilon$ people with $y=y_h$. As before,

$$(2) y_h = \frac{nm - n(1-\varepsilon)s}{n\varepsilon} = \frac{nm - ns + n\varepsilon s}{n\varepsilon} = \frac{m - s + \varepsilon s}{\varepsilon}$$

Substitute (2) and the rest into (1)

$$\begin{aligned} T^*(1) &= \frac{n(1-\varepsilon)}{n} \frac{s}{m} \ln \frac{s}{m} + \frac{n\varepsilon}{n} \frac{m-s+\varepsilon s}{\varepsilon m} \ln \frac{m-s+\varepsilon s}{\varepsilon m} = \\ &= (1-\varepsilon) \frac{1}{\alpha} \ln \frac{1}{\alpha} + \varepsilon \frac{m-s+\varepsilon s}{\varepsilon m} \ln \frac{m-s+\varepsilon s}{\varepsilon m} = (1-\varepsilon) \frac{1}{\alpha} \ln \frac{1}{\alpha} + \varepsilon \frac{\alpha s - s + \varepsilon s}{\varepsilon \alpha s} \ln \frac{\alpha s - s + \varepsilon s}{\varepsilon \alpha s} = \\ &= (1-\varepsilon) \frac{1}{\alpha} \ln \frac{1}{\alpha} + \frac{\alpha - 1 + \varepsilon}{\alpha} \ln \frac{\alpha - 1 + \varepsilon}{\varepsilon \alpha} \end{aligned}$$

lim T^* when $\varepsilon \rightarrow 0$,

$$T^*(1) = \frac{1}{\alpha} \ln \frac{1}{\alpha} + \frac{\alpha - 1}{\alpha} \ln \frac{\alpha - 1}{\varepsilon \alpha} = \frac{1}{\alpha} \ln \frac{1}{\alpha} + \frac{\alpha - 1}{\alpha} \ln \infty$$

So $T^*(1)$ tends to infinity. Note also that the first term will be negative because $1/\alpha < 1$, and thus $\ln(1/\alpha) < 0$.

Consider several numerical examples.

Let $\alpha=2$,

$$T^*(1) = \frac{1}{2} \ln \frac{1}{2} + \frac{1}{2} \ln \frac{1}{2\varepsilon} = A + B$$

When $\varepsilon=0.01$, then $T^* = A + 1.96$. When ε is smaller (0.001), $T^* = A + 3.11$, and if $\varepsilon = 0.0001$, then $T^* = A + 4.26$. So as ε decreases, T^* diverges rather than converges. That's the problem.

Let $\alpha=5$. With $\varepsilon=0.01$, $T^* = A + 3.51$. When ε is smaller (0.001), $T^* = A + 5.35$ etc. Again, it diverges. Thus, the maximum feasible Theil (1) as a function of α diverges.

Let $\alpha=10$. With $\varepsilon=0.01$, $T^* = A + 4.05$; with $\varepsilon=0.001$, $T^* = A + 6.12$ etc.

As can be seen in Table A1, the maximum feasible Theil (1) diverges as ε becomes smaller and tends toward 0. Thus, the inequality possibility frontier will be ∞ .

Table A1. Maximum T*(1) for different α and ε

$\alpha \backslash \varepsilon$	1/100	1/1000	1/10000
2	1.96	3.11	4.26
5	3.51	5.35	7.19
10	4.05	6.12	8.19

C. Derivation of the maximum standard deviation of logs for a given α

$$\sigma^* = \sqrt{\frac{1}{n} \sum (\ln y_i - \ln \mu)^2} = \left(\frac{1}{n} \sum (\ln y_i - \ln \mu)^2 \right)^{1/2}$$

We know as before that the income of the elite is:

$$y_h = \frac{nm - n(1 - \varepsilon)s}{\varepsilon n} = \frac{nm - ns + n\varepsilon s}{\varepsilon n} = \frac{m - s + \varepsilon s}{\varepsilon}$$

$$\sigma^* = \left(\frac{1}{n} n(1 - \varepsilon) (\ln s - \ln \mu)^2 + \frac{1}{n} \varepsilon n (\ln y_h - \ln \mu)^2 \right)^{1/2}$$

$$\sigma^* = \left((1 - \varepsilon) (\ln s - \ln \mu)^2 + \varepsilon \left(\ln \frac{\mu - s + \varepsilon s}{\varepsilon} - \ln \mu \right)^2 \right)^{1/2}$$

$$\sigma^* = \left((1 - \varepsilon) (-\ln \alpha)^2 + \varepsilon [\ln(\mu - s + \varepsilon s) - \ln \varepsilon - \ln \mu]^2 \right)^{1/2}$$

$$\sigma^* = \left((1 - \varepsilon) (\ln \alpha)^2 + \varepsilon [\ln(\mu - s + \varepsilon s) - \ln(\varepsilon \alpha)]^2 \right)^{1/2}$$

when $\varepsilon \rightarrow 0$, the second term in the previous expression tends to $0[\ln(\mu - s) + \infty]^2 = 0 \times \infty$.

Thus we have to find the limit of

$$(\varepsilon (\ln(\mu - s + \varepsilon s) - \ln(\varepsilon \alpha))^2)$$

when $\varepsilon \rightarrow 0$.

$$\frac{\varepsilon}{\frac{1}{(\ln A - \ln \varepsilon \alpha)^2}} = \frac{1}{-2(\ln A - \ln \varepsilon \alpha)^{-1} \left(\frac{1}{A} A' - \frac{1}{\varepsilon \alpha} s \alpha \right)} = \frac{\ln A - \ln \varepsilon \alpha}{-2 \left(\frac{1}{A} A' - \frac{1}{\varepsilon} \right)} = \frac{\infty}{\infty}$$

where $A = \mu - s + \varepsilon s$ and $\ln A = \text{constant}$ if $\varepsilon \rightarrow \infty$, also A'/A constant if $\varepsilon \rightarrow \infty$. Also $A' = s$.

Continue with L'Hôpital's rule when $\varepsilon \rightarrow 0$

$$\frac{\ln A - \ln \varepsilon s \alpha}{-2\left(\frac{1}{A}A' - \frac{1}{\varepsilon}\right)} = \frac{\frac{A'}{A} - \frac{s\alpha}{\varepsilon s \alpha}}{-2\left(-\frac{1}{A^2}s^2 + \frac{1}{\varepsilon^2}\right)} = \frac{\frac{1}{A\varepsilon}(s\varepsilon - A)}{\frac{2}{A^2\varepsilon^2}(s^2\varepsilon^2 - A^2)} = \frac{A\varepsilon(s\varepsilon - A)}{2(s^2\varepsilon^2 - A^2)} = \frac{0}{-2A^2} = 0$$

Therefore,

$$\lim(\varepsilon \rightarrow \infty) \sigma^* = \left((1 - \varepsilon)(\ln \alpha)^2\right)^{1/2} = \ln \alpha.$$

Thus, the maximum value taken by the standard deviation of logs when average income = α , is $\ln \alpha$.