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The foundations of numerical and mathematical abilities: A literature review

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Executive Summary

We constantly use numbers in our everyday lives, from estimating how long it will take us to get from A to B to being able to use numerical information to make financial decision. How do we acquire these numerical and mathematical skills? The present paper reviews evidence from studies with animals, infants, children and adults using both neuroscientific and behavioral methods (such as the measurement of how fast and accurate people are at solving problems) to study the foundations of number processing. The paper provides a detailed descriptions of the foundational systems of number processing that humans share with animals and discusses the role these play in early learning about number and calculation. The key conclusions and recommendations from this review are:

The foundations for number processing are laid long before the onset of formal schooling and are universal

One of the key implications that can be derived from the literature reviewed in the present paper is that the precursors to numerical and mathematical processing can be measured very early in life. Infants can discriminate between two arrays containing different number of dots and preschoolers can compare dot arrays for their relative magnitude. In other words, there is a universal foundation for number processing. Consequently there are several practical implications: 1. The foundations of numerical processing can be measured before children acquire symbolic representations of number (such as Arabic digits). Thus, individual differences in number processing can be measured very early on, paving the way towards strengthening early number skills and facilitating the acquisition of symbolic number processing. It is well known from other domains that early intervention and diagnosis of learning difficulties is key to prevent long-term failure. The results reviewed within this paper clearly show that it is possible to measure variability in early number processing.

Assessments of early number processing should include measures of symbolic and non-symbolic number processing

Numbers can be represented as symbols (i.e. '5' or 'five') or as non-symbolic representation (***** or /////). Many of the currently available measurement tools (some of which are reviewed in this paper) focus almost exclusively on symbolic representations of number, such as digits and number words. The present review clearly suggests that children can process numerical magnitude before they learn these symbolic systems. Therefore tests of number processing skills should contain non-symbolic items as well as tasks that measure children's ability to map between symbolic and non-symbolic representations of numerical magnitude.

Timed tests should be included in assessments of numerical and mathematical abilities

A further implication from the present literature review is that assessments of children's numerical and mathematical abilities should include timed measure (not exclusively, but timed measured should be a core component of assessment tools). In other words, tests should include time limits (which should be determined through rigorous pilot research). Speeded tests allow for the assessment of fluency. As is the case for reading, fluency is

key in numerical processing. Take the example of arithmetic. In order to quickly solve arithmetic problems, children need to process the numerical magnitude represented by the operands in a calculation problem. In other words, they need to be able to quickly activated and mentally manipulate the numerical magnitude represented by numerical symbols.

Assessments of number processing should be driven by peer-reviewed empirical evidence

In addition to the critical role played by timed tests in capturing individual differences in numerical and mathematical processing, the items that are included in assessment tools should be carefully selected. Here it is critical that test item selection is based on what is known about children's development of numerical and mathematical skills and the predictors of individual differences in these competencies. Too often tests of numerical and mathematical skill development do not have a clear model of learning and developmental change underlying them and therefore the justification for the inclusion of certain items is not always clear and neither is the number of items used to measure a particular construct. Such tests may fail to capture important individual differences. The present review strongly suggests that tests should be administered early and should include measures of numerical magnitude processing that have been demonstrated through peer-reviewed empirical research, such as dot counting, symbolic and non-symbolic comparison, number lines estimation, to be correlates and predictors of individual differences in children's numerical and mathematical skill acquisition. Too many tests measure what children are taught in school rather than the competencies that allow children to acquire those skills.

Helping children to understand the meaning of numerical symbols should be a core activity in preschool and early number

The literature review suggests that children's ability to use numerical symbols is a key predictor of numerical and mathematical skill acquisition. Therefore, preschool and early primary education should invest time in helping children gain a solid understanding of the meaning of numerical symbols and the relationships between symbolic and non-symbolic numerical magnitude representations as well as symbol-symbol relationships (numerical order). Activities such as matching games, activities involving number lines (such as Snakes and Ladders) may be particularly helpful here.

Introduction

The processing of numerical information is a dominant feature of everyday life. Whether it is in the context of financial transactions, such as shopping for groceries or in gauging how many people are in a room, we are constantly using numerical information to guide our behavior and make decisions. While there has been much more attention placed on literacy and its importance for life success, there is clear evidence to suggest that basic numerical and mathematical skills play a critical role in determining an individual's life success. Furthermore, there is abundant evidence that low numeracy skills are associated with substantial costs to society at large (Butterworth, Varma, & Laurillard; Bynner & Parsons, 1997; Duncan et al., 2007). In view of this, a better understanding of how the ability to process numerical information develops could lead to improvements in how these skills are taught and used by individuals to improve their lives across the globe. As an extension of this, improving numerical skills among individuals will have benefits for society at large as it will lead to a stronger working force and thus measurable economic improvements.

So what do we know about the mechanisms underlying our ability to process numbers? How do number processing competencies develop? What is the role of our brain in number processing? The aim the present literature review is to address these and related questions through an examination of the latest research on number processing from within the fields of cognitive psychology and neuroscience.

The focus of the review will be on the foundations of number processing abilities upon which higher-level abilities, such as mental arithmetic are built. In this way, the literature review does not provide a comprehensive overview of the development of numerical and mathematical skills, but rather explore the foundational systems underlying numerical and mathematical abilities. By doing so the review aims to identify key variables that characterize foundational competencies of numerical abilities that can be used in the context of educational assessments and become targets for educational interventions. It is a central contention of the present review that in order to improve the early acquisition of numerical competencies, a focus on foundational systems is necessary. If the foundations are well set then the system can develop optimally. Furthermore, in this paper I argue that focusing on foundational competencies can point to universal ways in which the early development of mathematical skills can be fostered.

Much of this focus is inspired by the tremendous progress that has been made in research on reading development. Specifically, in research on the acquisition of reading it has become apparent that if foundational competencies, such as phonological awareness, are not adequately acquired early on, then children will have a delayed developmental trajectory of learning how to read. This cumulative effect of early, foundational skills on later outcomes has been referred to as the 'Matthew Effect'(Stanovich, 1986). This finding highlights the importance of studying foundational systems in an effort to improve the starting points of skill acquisition, which have a dramatic, cumulative influence on later outcomes. In view of the evidence from reading, one of the key questions that will be explored below is whether the same is true for numerical and mathematical skills? In other words, do foundational competencies predict the acquisition of higher-level competencies such as mental arithmetic?

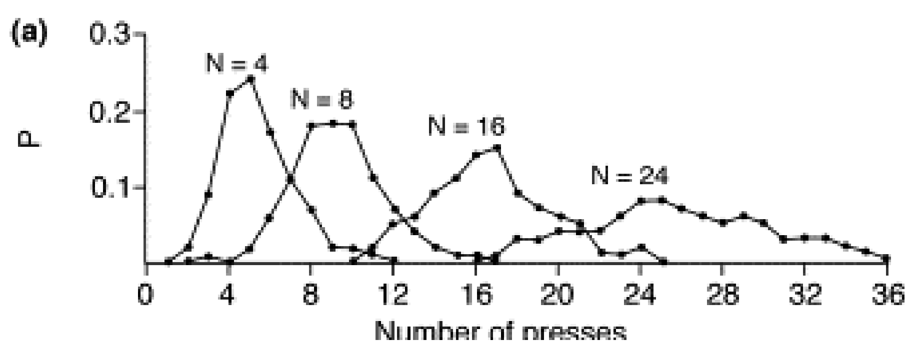
Before being able to address these questions, however, a detailed characterization of the foundational systems underlying number processing and how they change over developmental time is necessary.

A basic system for the representation and processing of numerical magnitude

While reading is a uniquely human skill, which is the product of cultural history and the invention of writing systems, non-human animals have been found to have basic number processing abilities and a lot of what we know about the foundations of our ability to represent and process numerical magnitude comes from the study of non-human animals. In particular, a large number of experiments have shown that animals can estimate and compare numerical magnitudes (total number of items in a set).

Specifically, in some of the earliest work on numerical magnitude processing abilities in animals, rats were trained to press a lever a certain number of times to receive a food reward. When the responses of these rats were plotted, the number of times the rats attempted to press the lever was approximately normally distributed around the required number of presses (Platt & Johnson, 1971). In other words, when the animals had to press the lever say 5 times, they most frequently responded with 5 presses, slightly less often with either 4 or 6 presses and even less frequently with 2 or 8 presses (leading to a quasi-Gaussian response profile around the required numerical magnitude). Furthermore, the larger the required number of presses, the greater the variability in the number of presses made by the animals (see Figure 1 below). Thus the larger the target number (the number of presses which would give the animals a food reward) the more often the animals pressed an incorrect number of times and the further their incorrect number of presses were away from the target number).

Figure 1. *Evidence for numerical magnitude representations in rats. Rats were trained to press a level a certain number of times. The figure shows the distribution of their responses around different target numbers.*

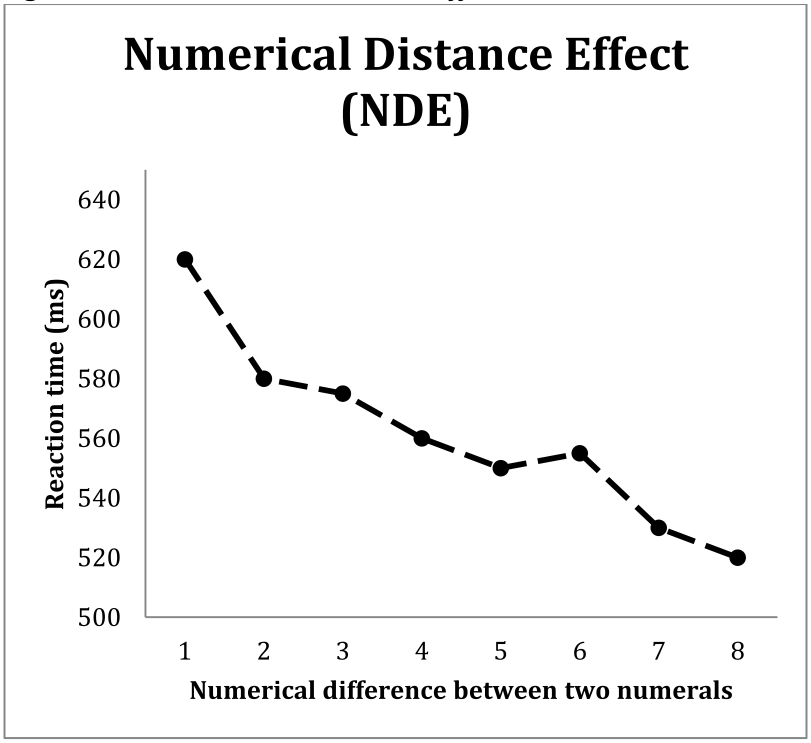


These observations provide some insights into how these animals are representing numerical magnitude. Specifically, they show that the animals' numerical magnitude representations are approximate, rather than exact. The distinction between exact and approximate representations of numerical magnitude refers to the difference between having a precise (i.e. there are exactly 50 white stars on the US flag, not 51 or 49, but exactly 50) and an imprecise, rough representation of number (i.e. there are between 200 and 400 billion stars in the milky way and it is not know exactly how many stars there are – it could be 402 billion or 486 billion or any number in between). From the available data it appears that humans share with animals the ability to represent numerical magnitude approximately, while exact representation seems to be uniquely human and may be tied to language. If animals were representing numerical magnitude exactly then we would expect little or no variability in their responses. However, as can be seen from the data plotted in Figure 1, rats vary in their responses when estimating magnitudes and their responses are roughly normally distributed around the desired numerosity. Moreover, the larger the numerical magnitude is, the less precise its representation. In other words, the variability in the animal's responses is proportional to the numerical magnitude (also referred to as 'Scalar Variability')

Interestingly, very similar responses were observed in human adults in an experiment that was similar to the method used to study numerical magnitude representations in rats. Specifically, adults were instructed to produce a number of key presses as quickly as they could without counting. Like the rats, their number of presses were normally distributed around the target number (they number of times they had been asked to press) and these distributions increase in proportion to the numerical magnitude of the target number (Whalen, Gallistel, & Gelman, 1999). These findings suggest that humans and non-human animals share an approximate system for the representation of numerical magnitudes.

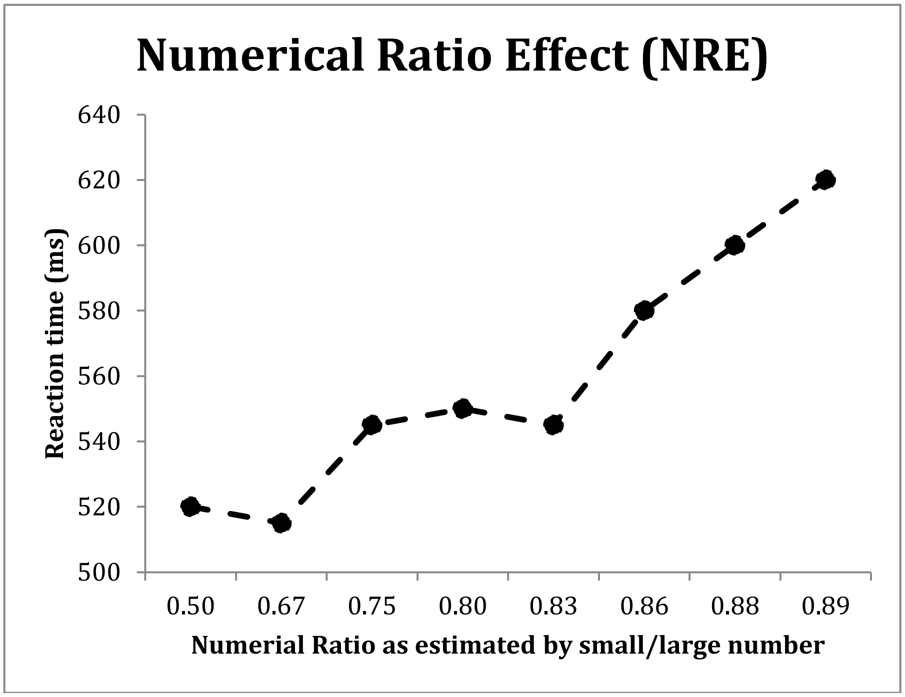
More evidence for a basic, approximate system for the representation of numerical magnitude comes from studies that have used number discrimination tasks. In a seminal study by (Moyer & Landauer, 1967) undergraduate students were asked to decide which of two simultaneously presented single digit Arabic numerals is numerically larger. When the researchers inspected how long it took students to decide which of the two numerals was numerically larger, they observed that it depended on the numerical difference (or distance) between the two numbers. Specifically, when the numerical distance between the numbers was relatively small (i.e. 1 or 2) participants were much slower and more likely to make an error compared to pairs of numbers that were separated by a comparatively large numerical difference (i.e. 8 or 9). This finding has become known as the 'Numerical Distance' effect (see Figure 2 for an illustration of this effect) and has been replicated in numerous studies since Moyer and Landauer's original report.

Figure 2. *The Numerical Distance Effect*



Another effect, that is complementary to the numerical distance effect is the so-called 'numerical ratio effect', sometimes also referred to as the 'numerical size effect' (See Figure 3). Say you are asked to compare 1 vs. 2 and 9. vs. 8. Both pairs have a numerical distance of 1 but their relative size differs. Specifically the ratio (smaller/larger) for 1 vs. 2 is 0.5, while for 9 vs. 8 it is 0.88. When researchers have compared the reaction times between such pairs of equivalent distance but different ratios it has been observed that it takes participants longer to decide which numeral is larger for pairs with relatively large numerical ratios. The ratio effect is consistent with Weber's Law, which states that the larger the magnitude of two stimuli that are being discriminated, the more difficulty the discrimination. In other words, the discriminability of two stimuli is proportional to their relative size.

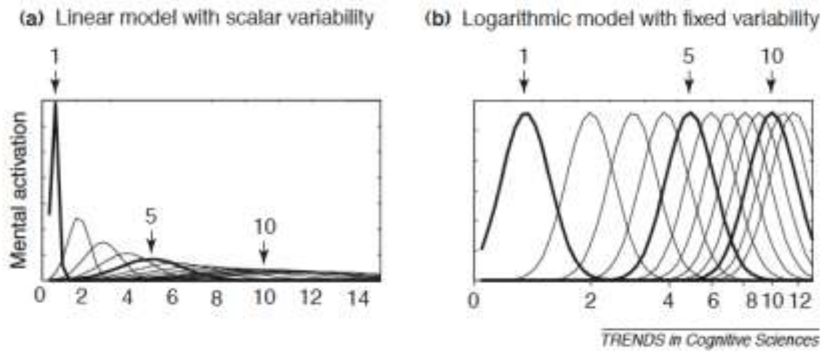
Figure 3. *The Numerical Ratio Effect*



The distance and ratio effect have become standard tools for assessing the representation and processing of numerical magnitude. Many researchers argue that the distance and ratio effects reflect an underlying system for the representation and processing of numerical magnitude. Specifically, it has been argued that these effects reflect the same system for numerical magnitude representation that had previously been uncovered through the experiments with rats, discussed above. The argument is that numbers that are closer together are represented more similarly than those that are far apart. In other words, the distance and ratio effects are thought to reflect an approximate, analog system of numerical magnitude representation.

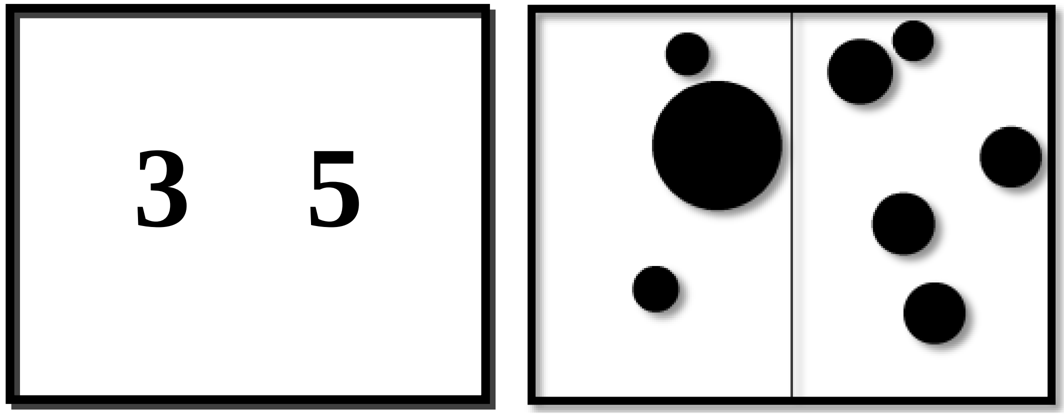
One way to think about this is to imagine that numbers are represented along some continuum that one might call a 'mental number line' (it should be noted that this is strictly metaphorical and does not imply a linear representation of numerical magnitude in the brain). Numerical magnitudes are placed on this 'mental number line' and each of these magnitudes has a representational distribution (such as a Gaussian Tuning Curve). In this way numerical magnitudes that are close on the number line have more overlap in their tuning curves, making them harder to tell apart, which leads to the numerical distance effect. The size effect can be accounted for either by increasing width of the tuning curves of individual magnitudes as a function of the size of the numerical magnitudes (as suggested in the data shown in Figure 1 above and Figure 4a below) or the spacing of the numerical magnitudes on the number line may decrease with increasing size (logarithmic compression, See Figure 4b).

Figure 4: *The figures shows two models of the way in which numerical magnitudes may be represented along a 'mental number' line. Both models can account for the numerical distance and ratio effects.*



A key distinction that is made in research studies on number processing is between **symbolic** and **non-symbolic** formats for the presentation of numerical magnitude. Symbolic representations refer to abstractions of numerical quantities such as digits, ideographs, number words or roman numerals. These symbols represent numerical quantities, but the relationship between symbols and quantities is arbitrary or non-iconic. Symbols become representations of numerical magnitude over the course of learning and development. In other words, there is nothing physical about the symbol 7 that suggests that this symbol represent the numerical magnitude 7. Non-symbolic representations, on the other hand, are iconic representations of numerical magnitude. Through counting or estimation the numerical magnitude of a non-symbolic set of items can be established. In contrast, the fact that '7' represents the numerical magnitude 7 is a cultural convention. It is important to note that the distance and ratio effects can be measured using both symbolic (e.g. Arabic numerals) and non-symbolic (e.g. arrays of dots) stimuli. Thus these effects can be measured in both individuals (or species) that do or do not have symbolic systems for the representation of numerical magnitude (such as Arabic numerals). This has important implications for the study of illiterate populations who may not be able to process symbolic representations of numerical magnitude.

Figure 5. *Example of stimuli for a symbolic and non-symbolic comparison task*



By using non-symbolic stimuli, researchers (Brannon & Terrace, 1998; Cantlon & Brannon, 2006) have been able to demonstrate that non-human primates also exhibit distance and ratio effects when they engage in the discrimination of pairs of non-symbolic magnitudes (such as discriminating between two groups of fruit). Thus the distance and ratio effects tap into a system for the representation of numerical magnitudes that can be found across species and thus appears to be a foundational system. A key question in the literature concerns the connection between non-symbolic and symbolic representations of numerical magnitude. I explore the question of how symbols acquire their meaning and how mappings between symbolic and non-symbolic representations of numerical magnitude might occur over developmental time in the below section entitled:" The development of numerical magnitude processing".

In summary, the data and theory reviewed in the above section demonstrates that humans and animals share a system for the representation and processing of numerical magnitude. Furthermore, this system can be clearly characterized with effects such as the distance and ratio effect (or using estimation tasks, such as finger tapping, key presses, or verbal estimation). These effects have lead researchers to suggest that our representation of numerical magnitude is approximate and noisy. Broadly speaking, the finding suggest that numbers are represented on a noisy mental number line, where numbers that are close together on the line overlap more than those that are relatively far apart. Moreover, as the size of the numerical magnitudes represented on the mental number line increases so does their noise, making relatively large numerical magnitudes harder to estimate and discriminate from one another than comparatively smaller numerical magnitudes.

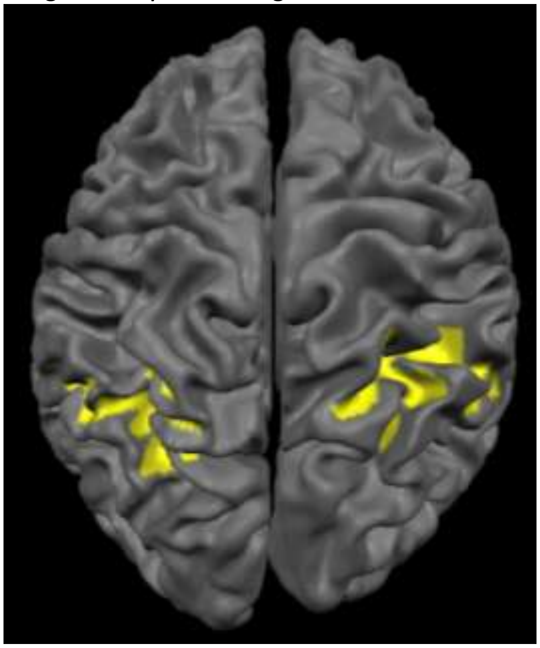
Importantly for educational consideration, the above reviewed literature suggest that a basic system for numerical magnitude representation has a long evolutionary history, which may suggest that, across cultures, mathematical learning starts with a universal basis upon which cultural systems of numerical and mathematical processing are built.

Numerical magnitudes in the brain

In addition to the behavioral evidence discussed above, much progress has been made in characterizing the neural circuits underlying numerical magnitude processing. Using non-invasive methods to measure brain function, such as Electroencephalography (EEG) and functional Magnetic Resonance Imaging (fMRI) neuroscientists have been able to measure the neural basis of basic numerical magnitude processing. These studies converge to suggest that regions of the left and right intraparietal sulcus (IPS) play a critical role in numerical magnitude processing (See Figure 6). For example, a large number of fMRI studies have shown that activity in this brain region is correlated with numerical distance (Ansari, Fugelsang, Dhital, & Venkatraman, 2006; Kaufmann et al., 2005; Pinel, Dehaene, Riviere, & LeBihan, 2001; Pinel et al., 1999). In other words when participants are asked to compare numerical magnitudes while their brain activation is measured, greater activation is found in the IPS while participants compare numerical magnitudes separated by a relatively small compared to large numerical distance. For comprehensive reviews of the neural correlates of numerical magnitude processing, please see (Ansari, 2008; Stanislas Dehaene, Piazza, Pinel, & Cohen, 2003; A. Nieder & Dehaene, 2009)

Furthermore, consistent with the behavioral evidence discussed above, there is also evidence for neuronal coding of numerical magnitude in non-human animals. By recording the activity of neurons in different brain regions while animals perform a particular task, a better understanding of how neurons encode information can be gained. Such single-cell recording techniques have also been applied to the study of numerical magnitude representations in animals. In a series of studies, Andreas Nieder and his colleagues have recorded from neurons in the parietal cortex while macaque monkeys perform a non-symbolic numerical magnitude discrimination task (Andreas Nieder, Freedman, & Miller, 2002; A. Nieder & Miller, 2003). What Nieder found was that in both regions of the prefrontal cortex and the IPS there are cells that preferentially fire when a certain non-symbolic numerosity is presented to the monkey. So for example there are cells that fire most vigorously when the monkey is presented with 5 dots. Furthemore, these cells code also respond to numbers that are close to their preferred numerosity and exhibit the numerical distance effect. Put different, a neuron that prefers 5 dots, also responds to the presentation of 4 and 6 dots, but slightly less than to 5 dots and even less when the monkey is presented with 2 and 8 dots. In other words, the further the numerical distance between the neuron's preferred numerical magnitude and the dot array that the monkey is being presented with the lower the firing rate. Thus the response patterns of single neurons exhibit response characteristics, such as approximate representations and numerical distance effects, that are consistent with the behavioral data reviewed above. These findings therefore suggest that numerical magnitude is encoded and processed at the single cell level in the brain and they exhibit the characteristic signatures of number representation, such as the distance and ratio effect discussed above.

Figure 6 *View of the human brain from the top. Highlighted in yellow is a region called the intraparietal sulcus (IPS) that has been found to be engaged during numerical magnitude processing.*



Having reviewed both the behavioral and neuroscientific literature on how numerical magnitudes are represented, I now turn to a review of the developmental literature to explore whether, and if so how, development and learning are associated with changes in the representation and processing of numerical magnitude.

Numerical magnitude processing in infants

In view of the evidence reviewed above that demonstrates qualitatively similar representations of numerical magnitude in non-human animals and human beings, it can be hypothesized that the representations of numerical magnitude might be innate and therefore detectable early in infancy. Indeed, there is a large and growing body of evidence demonstrating that very young human infants as well as newborns are sensitive to numerical magnitude. In such studies, infants are typically presented with a series of slides that display a certain number of objects. Throughout the experiment the infant's looking time (or in very young babies sucking time) is measured. After a certain number of repetitions of the same non-symbolic numerical magnitude, the infant will start looking away from the display. This is called habituation. The infant has become habituated to the repeated display of a particular numerical magnitude. When this happens, the experimenter will present the infant with alternating displays of 1. a new, different non-symbolic numerical magnitude and 2. the old numerical magnitude that the infant became habituated too. What the researcher wants to uncover in such an experiment is whether infants look longer at the new compared to the old non-symbolic numerical magnitude. If infants do indeed look longer at the new compared to the old non-symbolic numerical magnitude then they can said to have noticed a change in the number of objects displayed. In the early experiments of this nature, infants were found to be able to discriminate between small numbers of dots, such as two and three dots (Antell & Keating, 1983; Starkey & Cooper, 1980). Moreover infants were also found to be sensitive to simple arithmetic transformations, such as adding one doll to another doll or removing one doll from a display of two dolls (Wynn, 1992).

More recently, several investigations have revealed that infants as young as 6 month of age can discriminate between 8 and 16 dots, while at the same time not exhibiting a sensitivity to the difference between 8 and 12 dots (Xu & Spelke, 2000; Xu, Spelke, & Goddard, 2005). These findings not only show that young, preverbal infants can process numerical magnitude, but also provide evidence to suggest that infants are sensitive to numerical distance and numerical ratio in the same way as adults and non-human animals are (see above). Specifically, the infants can only discriminate between non-symbolic numerical magnitudes when these are sufficiently different in distance or ratio. Interestingly, the ratio at which infants succeed to discriminate between numerical magnitude changes over developmental time. For example (Lipton & Spelke, 2003), using an auditory number discrimination paradigm (infants discriminated between different number of sounds) were able to show that 6-month old infants can discriminate 16 from 8 sounds, but fail at discriminating between 12 and 8 sounds. By the time infants are 9-month old they can discriminate 12 from 8 sounds, but ail at discriminating between 10 and 8 sounds. These data suggests that the precision (or acuity) of the approximate representation of numerical magnitude increases over the early months of life.

Taken together there is a growing body of evidence (for a review see Libertus & Brannon, 2009) to suggest that preverbal infants are sensitive to non-symbolic numerical magnitudes. Moreover, the looking time data indicate that, as we saw from the adults and non-human animal data, discussed above, the numerical ratio and distance between numerical magnitude predicts the precision of numerical magnitude discrimination. Therefore, these data suggest the existence of a system for the representation and processing of numerical magnitude that is qualitatively similar in non-human animals, human adults and children. This strengthens the notion that we come equipped to life with foundational competencies for the processing of numerical information and that education has to build on these competencies in order to allow children to learn higher-level, cultural mathematical skills such as mental arithmetic.

However, while the data do suggest the existence of qualitatively similar systems for numerical magnitude representation in infants, animals and adults, the results from the study of infants also suggests that the acuity with which they can discriminate numbers is significantly lower than the precision of the representational system in adults. It may be helpful to think back to the metaphor of a ‘mental number line’ where each number has a certain representational distribution (or tuning curve). Given the data from infants we imagine that the tuning curves of adjacent numerical magnitudes on the number line are much wider, leading to greater overlap and thus more difficult discrimination. Thus there is evidence for developmental changes in the representation and processing of numerical magnitude. It is to a detailed consideration of these developmental changes and their implications that I turn to next.

The development of numerical magnitude processing

How can we measure the numerical magnitude processing in young children? Do researchers see any developmental changes? How do children go from having a non-symbolic sense of numerical magnitude to understanding that abstract symbols, such as Arabic numerals, represent numerical magnitudes? The aim of the next section of this paper is to address these interrelated questions through a review of the available empirical evidence.

The Development of Counting

One of the most significant events in early number development is children's development of counting skills. And while the focus of this review is on non-verbal representations of numerical magnitude, no review of the children's development of numerical skills should ignore the development of counting skills.

While early theories suggested that children's understanding of the meaning of counting (the knowledge that counting determines the total number of items in a set) is innate (Gelman & Gallistel, 1978), more recent evidence suggest that it is not until around the age of 3 ½ years that children understand the meaning of counting (Le Corre & Carey, 2007; Wynn, 1990). When children understand that the purpose of counting is to determine the total number of items in a set they can be said to have acquired the 'cardinality principle'. A popular task used to measure whether or not children have acquired the cardinality principle is the 'give-a-number task'. In this experimental task, children are given a pile of objects and asked to give different number of objects to an experimenter. Children who have acquired the cardinality principle will use counting to give the experimenter the exact number of objects requested. However, children who do not have an understanding of the meaning of counting will typically just grab a number of objects and give them to the experimenter without counting. These 'grabbers' have not yet understood the meaning of counting. What is fascinating, however, is that children who are 'grabbers' and therefore do not yet understand the cardinality principle, are often very apt at reciting the count sequence. For these children the count sequence is a meaningless verbal sequence, whose order they have learnt and are able to recite, but the number words they recite do not yet have a quantitative meaning for these children. The difference between a children who can count but does not understand the meaning of the activity and a children who can both count and knows that counting gets you quantity is an important one for both research and practice. Often parents will misunderstand good verbal counting skills with an understanding of how counting is used to enumerate.

Counting is widely regarded as the first step towards children's exact, verbal understanding of number. Counting is exact enumeration and thus different from the approximate number system described above. There currently exists controversy over whether or not the system for the representation of approximate numerical magnitude (discussed above) influences the acquisition of children's understanding of the cardinality principle (Le Corre & Carey, 2007; Wagner & Johnson). Whether or not approximate representation and processing of numerical magnitude plays a role in children's development of counting, the development of counting is a step towards exact, verbal number representation. Given the focus of the present review on the role played by the foundational, non-verbal system for the approximate processing numerical magnitude, the paper will now turn to developmental studies of such processes.

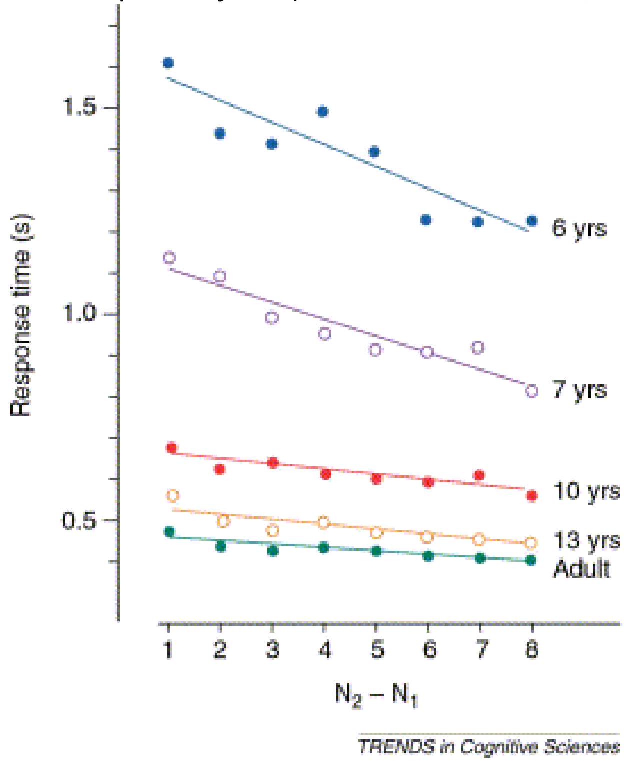
Developmental changes in numerical magnitude representations - evidence from comparison tasks

As discussed above, even before children learn how to count verbally, they exhibit sensitivity to numerical magnitude. In other words, there are basic mechanisms for the processing and representation of non-symbolic numerical magnitude that can be measured very early on in development. Specifically, the data from studies of preverbal infants have demonstrated that infants as young as 6 month of age can discriminate between non-symbolic numerical magnitudes (e.g. between 8 and 16 dots). Furthermore, as is the case for non-human animals and human adults, the ability of infants to discriminate between numerical magnitudes is dependent on their ratio and distance. As was reviewed above, the acuity of infant's non-symbolic numerical magnitude discrimination abilities changes over a period of months with 9 months old being able to discriminate larger (more difficult) ratios than 6-month old infants.

Such evidence raise questions concerning what changes in the processing and representation occur over the course of development? Moreover, young children become exposed to cultural, symbolic representations of numerical magnitude over the course of learning and development. How do children process symbolic representations of numerical magnitude, such as Arabic numerals and how does the mapping between symbolic and non-symbolic numerical magnitude representations occur?

With respect to developmental changes in basic numerical magnitude processing, there are a number of studies that demonstrate that numerical magnitude discrimination in young children is ratio and/or distance dependent (Halberda & Feigenson, 2008; Huntley-Fenner & Cannon, 2000) In other words, when young children compare which of two arrays of objects is numerically larger their accuracy and reaction times are related to the ratio and distance between the magnitudes they compare. Moreover, the strength with which numerical distance or ratio affect children's responses (accuracy and reaction time) changes over developmental time. More specifically younger children show greater effects of numerical ratio and distance on the accuracy and reaction time than older children. In other words, the difference in reaction times and accuracy between comparing numbers that are separated by a relatively small distance (large ratio) and those that have a relatively large distance (small ratio) is greater for young children compared to their older peers (See Figure 6 for an illustration of this developmental change in the numerical distance effect).

Figure 6 *Developmental Changes in the Numerical Distance Effect among literate children. Data reprinted from (Sekuler & Mierkiewicz, 1977)*



It has been proposed that the developmental changes in the numerical distance (or ratio) effect are reflective of changes in the representation and processing of numerical magnitude. More specifically, it has been contended that, over developmental time, approximate representations of numerical magnitude become less overlapping and fuzzy, allowing for faster access and greater ability to discriminate between numerical magnitudes that are close on the mental number line. If we think back to the metaphor of a 'mental number 'on which each number is represented with a certain representational distribution (tuning curve) then one might speculate that development involves a sharpening of these tuning curves, which leads to less overlap in the representational features of numerical magnitudes that are represented in close proximity to one another on the mental number line.

Alternative theories contend that the numerical distance and ratio effects reflect the comparison process rather than providing an index of the representation (Van Opstal, Gevers, De Moor, & Verguts, 2008; Zorzi & Butterworth, 1999). In this conceptualization, developmental changes in the distance and ratio effects may represent changes in the comparison process rather than the number line representation itself. In other words, the developmental differences observed reflect changes in the ability of children to relate numerical magnitude representations to one another in the context of a numerical magnitude comparison task (one might refer to this process as 'relational processing of numerical magnitudes'). Further research will provide more insight into the precise mechanisms underlying the developmental changes in basic numerical magnitude processing. For the purpose of the current review, the important observation is that the basic processing of numerical magnitude processing changes over developmental time and may therefore provide an important basis for school-relevant, higher-level numerical and mathematical achievement (see section entitled “Numerical magnitude processing and school-relevant mathematics achievement” below for more details).

Symbolic and non-symbolic numerical magnitude comparison

As discussed above and illustrated in Figure 5, numerical magnitudes can be represented both symbolically and non-symbolically. While non-symbolic representations of numerical magnitude can be measured in pre-verbal infants, symbolic representations are the outcome of learning and development. One of the large open questions in the field is when children learn the meaning of visual numerical symbols, such as Arabic numerals. Many researchers have hypothesized that symbolic representations are mapped onto pre-existing non-symbolic representations of numerical magnitude (Piazza, Pinel, Le Bihan, & Dehaene, 2007; Verguts & Fias, 2004). However, it is currently not know how this process occurs. Furthermore, there is some evidence to suggest that symbolic and non-symbolic representations are unrelated to one another. More specifically, studies with both children (Holloway & Ansari, 2009) and adults (Maloney, Risko, Preston, Ansari, & Fugelsang) failed to find significant correlations between symbolic and non-symbolic distance effects. Furthermore, the role of environmental factors (such as parental and/or community) input in this developmental process is currently not well understood.

Symbolic and non-symbolic numerical magnitude processing in illiterate participants

It is important to note that almost all of the studies discussed in this literature review have been conducted with literate adults and children who attend mainstream school. A largely under-investigated question concerns the symbolic and non-symbolic numerical magnitude processing competencies of illiterate participants. A number of studies with adults from remote communities in the Amazon who lack a verbal code for number (only have count words up to 3-4) indicates that despite having no symbolic representations of numerical magnitude, these individuals can nevertheless process non-symbolic numerical magnitudes (Gordon, 2004; Pica, Lemer, Izard, & Dehaene, 2004). In a recent study (Zebian & Ansari, 2012), Dr. Samar Zebian from the Lebanese American University in Beirut and I have compared the ability of illiterate and literate migrant workers in Syria to compare symbolic (Arabic numerals) and non-symbolic (arrays of squares) numerical magnitudes. Specifically, we measured their numerical distance effects for symbolic and non-symbolic comparisons. We found that for non-symbolic numerical comparisons the distance effect did not differ between illiterate and literate participants. However, for symbolic number comparison the illiterates (who could recognize Arabic numerals) showed a significantly larger numerical distance effect the literate participants. These findings thereby suggest that literacy affects symbolic but not non-symbolic numerical magnitude processing. This has important implications for education as it shows that education modulates the semantic representation of symbolic but not non-symbolic numerical magnitudes. It is important to note that, in this study, the literate and illiterate subjects were matched in terms of their profession (they were all Syrian migrant workers in Lebanon), so the only difference between the groups was their level of literacy and years spent in formal education (the literates self-reported having attending school for at least 10 years, while the illiterates reported to have, on average, spent 1 year or less in formal education).

Clearly there is a lot more work to be done on illiterate numerical cognition. In addition to more work with adults, there need to be research studies looking at the development of symbolic and non-symbolic numerical magnitude processing in children who are not in mainstream education and become illiterates. However, the above data suggest, consistent with the data from animals and preverbal infants, that non-symbolic numerical magnitude representations are foundational and qualitatively similar in educated and uneducated individuals. This might then also suggest that difficulties in number processing arise from the mappings between numerical symbols and the numerical magnitudes they represent. This has practical implications and suggests that spending time early in development teaching children about the meaning of numerical symbols by pairing them with non-symbolic numerical magnitudes (i.e. arrays of dots) might be particularly productive. This could take the form of matching games where children have to match symbolic and non-symbolic representations of numerical magnitude.

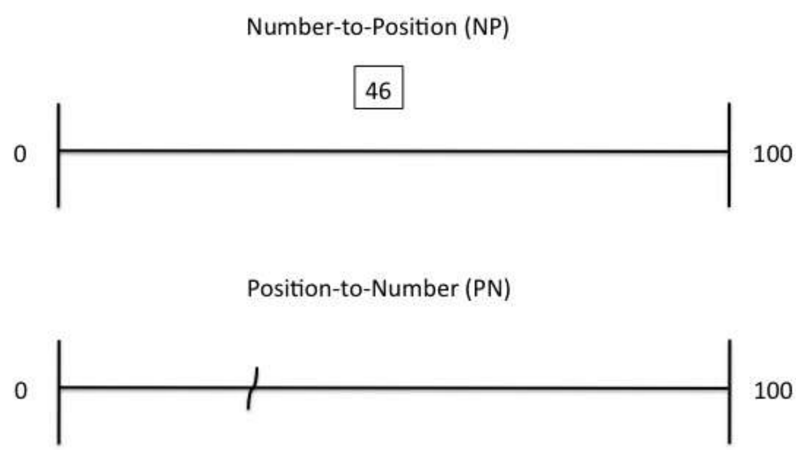
Developmental changes in numerical magnitude representations - evidence from estimation tasks

In addition to numerical magnitude comparison tasks, numerical estimation has been used to gain insights into numerical magnitude processing. Recall, for example, the experiment in which rats were trained to press a lever a certain number of times in order to obtain a food reward (Platt & Johnson, 1971). In essence the rats were performing an estimation task and their response profiles suggested that the variability in their estimates was proportional to the number they were required to estimate (See Figure 1 above). Estimation provides a measure of the variability (noise) in the representation and processing of numerical magnitude. In view of this estimation is also a paradigm that can be used to investigate developmental changes in the systems underlying numerical magnitude processing and representation. To do this, (Huntley-Fenner, 2001) asked 5-7 year olds to verbally estimate the numerical magnitudes of rapidly presented arrays of 5,7, 9 and 11 squares without counting. The results from this study revealed that children, like rats, exhibit variability in their estimations that are proportional to the numerical magnitude that they are asked to estimate. In other words, children were much more accurate and less variable when estimating 5 squares than they were at estimating 11 squares. Thus the estimation performance young children carries the same performance signature that has previously between observed in animals and provides further evidence in support of the existence of a basic system for the representation and processing of numerical magnitude that is qualitatively similar across species and over developmental time. With regards to the latter, Huntley-Fenner (2001) did find that the variability in children's estimates of the non-symbolic arrays decreased with ages. These data, like the developmental investigations of numerical magnitude comparison discussed above, do suggest that the acuity (or precision) of numerical magnitude processing increases over developmental time. In other words, with age children's ability to process numerical magnitudes becomes increasingly efficient, allowing them to access and process these representations more precisely and rapidly.

Another way to measure developmental changes in children's numerical magnitude estimation abilities that has gained increasing popularity in recent years is number-line estimation (Siegler & Opfer, 2003). Typically, in experiments exploring number line estimation participants will be presented with a line with zero at one end and an upper limit number, often 10, 100, or 1000, at the other end (see Figure 7). Individuals are asked to estimate where on the line a certain number falls (Number to Position Estimation; NP) or estimate what number is represented by a mark on the line (Position to Number Estimation, PN).

These experiments do not merely allow researchers to study developmental changes in the accuracy with which children can estimate symbolic representations of numerical magnitude (e.g. Arabic numerals) but also help them to understand how the relationship between numerical magnitude and position on the number line changes over developmental time and thereby compare difference in the way in which younger and older children map numerical magnitudes into space (in this case onto a number line).

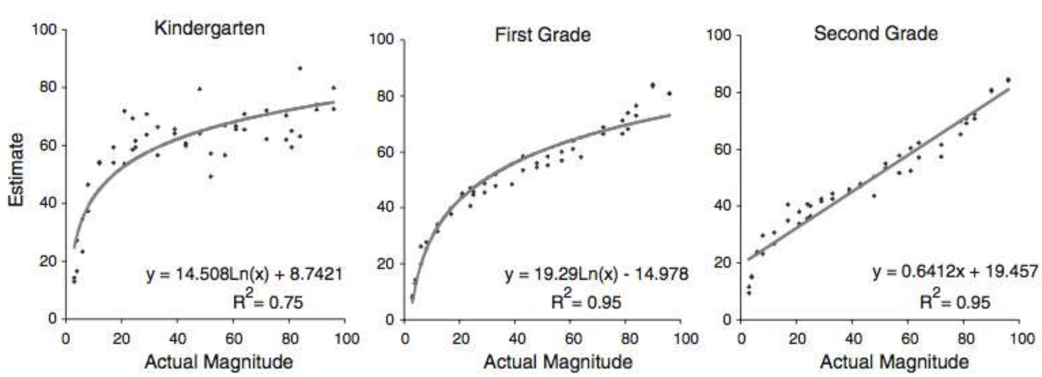
Figure 7 *Number Line Estimation task. In the Number-to-Position task, children are shown a number just above the number line and have to estimate its position. In the Position-to-Number task they are shown a hatch mark and have to estimate which number corresponds to the marked position on the number line.*



Recall from above (and see Figure 4 above) that there are two competing models for how numerical magnitudes may be internally represented: linear or logarithmic. It turns out that number line estimation is a very good way to distinguish between these different representations of numerical magnitudes along the number line. For instance, if an individual's representation of numerical magnitude is linear, one would expect their estimations to be equally spaced, regardless of the size of the numerical magnitude that is currently being estimated. In other words the difference between estimating 5 and 10 should be approximately the same as the difference between estimating the position on the number line for values 65 and 70. However, if their representation of number is logarithmic, one would expect her estimations to be more compressed on the right hand side of the line (where the larger numerical magnitudes are represented). For instance, estimates of smaller number could be spaced further apart and larger numbers spaced closer together. Therefore, in a logarithmic representation the difference in between number line estimates for 5 and 10 should be larger than the difference in estimations on the number line for 65 and 70. One can then use number line estimation to see how these representations and their variability change over developmental time and analyze what functional significance, if any, is associated with these changes.

Accuracy of number line estimation increases over developmental time. This trend has been observed for number lines of different sizes like 0-10 (Petitto, 1990), 0-100 (Petitto, 1990; Siegler & Booth, 2004b), and 0-1000 (Siegler & Opfer, 2003). In addition to overall accuracy, children's number representation has also been shown to undergo age-related changes. In general, changes in estimation reflect a shift in representation from logarithmic in younger children to linear in older children and adults (Siegler & Booth, 2004b; Siegler & Opfer, 2003). This shift has several salient characteristics. First, this developmental change is gradual and is punctuated by a period during which children's estimates are equally well described by both linear and logarithmic functions. Second, the shift occurs later in development for larger number lines. While the estimates of second graders are better characterized by a linear representation when making estimates on a 0-100 number line (Siegler & Booth, 2004b), the same age group's estimates on a 0-1,000 number line are better characterized by a logarithmic representation.

Figure 8 *Data from Siegler & Booth (2004) showing age related changes in number line estimation between kindergarten and first grade. The data show how number line estimates are initially logarithmic and become increasingly linear.*



What drives the change from logarithmic to linear representations of numerical magnitude? It may be that these changes are simply due to the maturation of the brain or, alternatively, these changes may be the product of the interaction between maturational changes and children's experience with numerical magnitude in the context of learning and education.

Two recent cross-cultural studies support the hypothesis that the shift from logarithmic to linear representations of numerical magnitude in number line estimation is the product of education. In the first study, (Dehaene, Izard, Spelke, & Pica, 2008) tested the performance of illiterate members of an Amazonian indigene group: the Munduruku. In previous studies of this population (Pica et al., 2004) it was shown that these individuals have a very limited number word vocabulary (most of them do not have number words for numerical magnitudes greater than 3). At the same time these individuals can perform numerical magnitude comparison tasks and exhibit the numerical ratio effect on non-symbolic numerical magnitude comparison tasks (e.g. judging which of two clouds of dots contains more dots) just like individuals who have full number word vocabulary. When tested on number line estimation using both symbolic (Arabic numerals) and non-symbolic (arrays of dots) endpoints, Munduruku participants of different ages all exhibited a logarithmic mapping of symbolic and non-symbolic numerical magnitudes onto a number line. Furthermore, for those Mundurku participants who had received some education in Portuguese language, the mapping of Portuguese numbers onto the number line was linear. Furthermore, analyses indicated that the degree of education predicted the linearity of participant's number line estimates. Specifically, individuals with relatively greater levels of education exhibited more linear number line estimates. This suggests that a logarithmic mapping of numbers onto number lines is a fundamental, or as the authors refer to it 'intuitive' representation of number line. Moreover, and crucial for the considerations of the current review, education was found to modulate the linearity of the number line estimates with more linear estimates among the relatively more educated Munduruku participants. Taken together, in the absence of formal education, the mapping of numerical magnitudes into space is logarithmic and becomes linear through the process of formal education. A linear representation is essential for mental arithmetic and exact number processing.

More evidence to suggest that it is education and the quality thereof that influences the patterns of number line estimation comes from a cross-cultural study comparing the number line estimation performance in kindergartners in China and the United States (US). This comparison revealed that the number line estimate of Chinese kindergartners are linear significantly earlier than is the case for their US peers (Siegler & Mu, 2008). In the group of American children, 88% of children were Caucasian and 12% Asian. It is well known that Chinese children outperform their US peers on many formal tests of mathematical abilities (Stevenson et al., 1990). The number line data from Siegler and Mu provide evidence to suggest that these differences can be detected before children enter their respective, formal educational settings. By doing so, the data suggest that cross-cultural differences in preschool numerical activities in China and the US have a profound effect on children's representation of numerical magnitude. Like the data from the Munduruku, these findings demonstrate that cultural environment and education (both formal and informal) affect numerical magnitude representations and mediate the shift from intuitive (logarithmic) to formal (linear) mappings of numerical magnitudes onto number lines.

Taken together, number line estimation provides insights into changes in numerical magnitude representations over the course of learning and development. Specifically, the shift from logarithmic to linear representations reflects a shift away from basic, intuitive representations of numerical magnitude to enculturated, formal representations of numerical magnitude that aid the learning of higher-level numerical and mathematical skills, such as mental arithmetic. In view of this number line estimation can be used to track and characterize individual differences in children's representation of numerical magnitude and can distinguish between children who are reliant on the basic representation of numerical magnitude that are shared across species and develop early and those children who, through the process of learning and education have 'enriched' their representations and rely on linear representations of numerical magnitude.

Developmental changes in the brain representation of numerical magnitude

Besides using behavioral methods such as numerical magnitude comparison and estimation, it is possible to use non-invasive neuroimaging methods such as functional Magnetic Resonance Imaging (fMRI) to investigate the effects of learning and development on the neural correlates of numerical magnitude processing. There are now several published fMRI studies that have revealed that the neural correlates of numerical magnitude processing change over developmental time (Ansari & Dhital, 2006; Ansari, Garcia, Lucas, Hamon, & Dhital, 2005; Cantlon, Brannon, Carter, & Pelphrey, 2006; Cantlon et al., 2009; Holloway & Ansari, 2010). These studies, broadly speaking, converge to suggest that areas of the intraparietal sulcus (IPS) become specialized for the processing and representation of numerical magnitude over developmental time. As discussed above, the IPS plays a critical role in the processing and representation in the brains of human adults and non-human animals. The developmental data suggests that this region assumes its role in numerical magnitude processing over the course of learning and development.

In addition, several of the developmental studies cited above, have revealed that children, to a greater extent than adults, recruit prefrontal brain regions during numerical magnitude comparison tasks. For example, Ansari et al. (2005) found that children exhibited a distance effect on areas of the right prefrontal cortex while adults showed distance related modulations of activation in the IPS. Thus children appear to use areas involved in attention, working memory and cognitive control during numerical magnitude processing to a greater extent than do adults. One interpretation of these data is as follows: in children, consistent with the behavioral data reviewed above, the representations of numerical magnitude are more overlapping, fuzzy and approximate compared to adults. Therefore, in order to resolve these representations during numerical magnitude comparison tasks (such as judging which of two Arabic numerals is numerically larger) children require frontal lobe systems to tease apart parietally-mediated representations of numerical magnitude. Furthermore, it has been speculated that the prefrontal cortex plays a critical role in the mapping between symbolic and non-symbolic numerical magnitudes (Cantlon et al., 2009; Diester & Nieder, 2007; A. Nieder, 2009). Thus it may the case that greater prefrontal activation in children may be indicative of, compared to adults, less fully developed processing of the semantic (numerical magnitude) meaning of numerical symbols.

Further evidence in support of the notion that the parietal cortex undergoes a process of specialization for the processing and representation of numerical magnitude comes from the study of children with specific difficulties in mathematical skills. Children who have mathematical difficulties in the absence of general impairments of learning and intellectual functioning are often referred to as exhibiting 'Developmental Dyscalculia'. In one of the first functional neuroimaging studies of Developmental Dyscalculia (DD) , Price et al. (2007) found that children with DD activate their IPS much less than typically developing children (those without mathematical difficulties) during non-symbolic numerical magnitude comparison (Price, Holloway, Rasanen, Vesterinen, & Ansari, 2007). More specifically, while children without DD exhibited a distance effect on activation of the parietal cortex during non-symbolic numerical magnitude comparison (greater activation for relatively small compared to larger numerical distances), children with DD exhibited overall lower activation of they IPS, compared to their typically developing peers, and no distance related activation of the IPS.

Taken together, the evidence from the available neuroimaging evidence is consistent with the behavioral data in suggesting that there are significant development changes in the processing and representation of numerical magnitude. Moreover, these findings reveal that brain regions that are known to play a critical role in numerical magnitude processing in adult participants, undergo a process of age-related specialization and that this process of specialization is disrupted in children with mathematical difficulties, such as Developmental Dyscalculia.

Numerical magnitude processing and school-relevant mathematics achievement

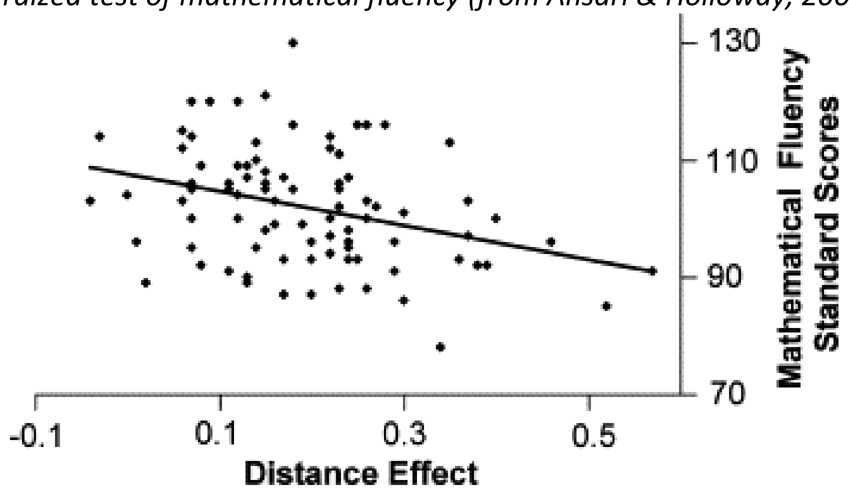
The developmental literature reviewed above demonstrates that there are significant age-related changes in the basic representation and processing of both symbolic and non-symbolic numerical magnitude. Data from studies of numerical magnitude comparison, estimation and number line estimation all converge to suggest that the representation and processing of numerical magnitude becomes increasingly precise over developmental time or, put different, the acuity (or resolution) of numerical magnitude representation undergoes age-related improvements. From an educational perspective, this raises a number of questions, such as: Are individual differences in measures of numerical magnitude processing, such as numerical magnitude comparison or estimation related to variability in children's mathematical abilities? How specific are these relationships? Can measures of numerical magnitude processing be used in the context of educational testing?

The above and related questions have recently been addressed in a number of cross-sectional and longitudinal studies. In one of the first studies to investigate whether individual differences in measures of numerical magnitude processing are related to individual differences in children's mathematical achievement, Halberda et al. (2008) tested the non-symbolic number discrimination abilities of a group of 14 year old's whose mathematical development had been followed since kindergarten. Specifically, the teenager had to judge which of two arrays (one yellow one blue) was numerically larger. On the basis of the accuracy with which the participants were able to compare the dot clouds of different ratios, the researcher computed a measure of the acuity with which children were able to compare the dot arrays (called 'Weber Fraction'). The Weber Fraction is a measure of the precision with which individuals compare numerical magnitudes and is thought to index how sharply numerical magnitudes are represented in the brain. Correlations between this measure of number acuity and individual differences in children's math scores measured in kindergarten and throughout elementary school revealed that those children who had more precise representations of non-symbolic numerical magnitude at age 14 had also performed better on various standardized tests of mathematical skills measured when these children were younger. This finding therefore suggests that there is a relationship between individual differences in numerical magnitude representations and individual levels of mathematical skill development. Moreover, the relationships between the teenager's non-symbolic numerical magnitude discrimination abilities and their math scores measured when they were younger was found to be rather specific, as these relationships were still significant after controlling for variables such as the children's working memory scores, language skills and intelligence.

Thus these findings suggest a very robust and specific relationship between individual differences in the precision of non-symbolic numerical magnitude representations and scores on standardized tests of mathematical skills. In other words, the results suggest that individual differences basic, intuitive representations of numerical magnitude related to symbolic numerical and mathematical processing skills. The findings are somewhat surprising because individual differences in non-symbolic number discrimination were measured at age 14 and related to variability in the participant's mathematical skills measured earlier in life. From these findings, the causal relationship between individual differences in the acuity of non-symbolic numerical magnitude representations and math achievement cannot be unambiguously been established. On the one hand it is possible that early-developing individual differences in the precision of non-symbolic numerical magnitude processing lead to individual differences in school-relevant, symbolic numerical and mathematical skills. However, because the correlations were retrospective it is also possible that individual differences in mathematical skill acquisition lead to changes in the acuity of non-symbolic numerical magnitude processing and representation. To resolve these interpretations prospective, longitudinal studies are necessary.

In addition to the above data providing evidence for a relationship between non-symbolic numerical magnitude processing and mathematical achievement, there are a number of published studies that have revealed that individual differences in symbolic number comparison are both correlated and predictive of individual differences in children's arithmetic achievement. For example, (Holloway & Ansari, 2009) tested the numerical distance effect in 1st and 2nd grade children in the United States and correlated individual differences in both the symbolic (Arabic numerals) and non-symbolic (arrays of squares) distance effect with the children's scores on standardizes test of mathematical fluency and calculation. The authors found a negative correlation (see Figure 9 below) between the size of an individual's numerical distance effect and their score on standardized tests of mathematical achievement. In other words, those children whose response times during number comparison were relatively more affected by numerical distance, resulting in a larger distance effect, had comparatively lower scores on standardized tests of mathematical achievement compared to their peers who had relatively smaller distance effect. This correlational effect was found to be specific as no significant correlations between individual differences in the size of the numerical distance and reading achievement scores were found. Recall, from the review of the developmental evidence of number comparison above that the numerical distance effect decreases over the course of learning and development. Thus a larger numerical distance effect may reflect a more immature and less precise representation of numerical magnitude. Thereby these correlational data converge with Halberda et al's (2008) data by suggesting that individuals with coarser representations of numerical magnitude have poorer math skills. Interestingly, Holloway & Ansari (2009) found correlations between the symbolic, but not the non-symbolic distance effect and children's math scores.

Figure 9 *Negative correlation between individual differences in 1st and 2nd grade children's symbolic numerical distance effect and their scores on the Woodcock Johnson standardized test of mathematical fluency (from Ansari & Holloway, 2009)*



Correlational evidence consistent with Holloway & Ansari's (2009) finding has been reported in a number of other studies using both measures of the numerical distance as well as the numerical ratio effect (Bugden & Ansari, 2011; Mundy & Gilmore, 2009). Moreover, in a longitudinal study of children in Belgium, (De Smedt, Verschaffel, & Ghesquiere, 2009) found that individual difference in children's numerical distance effect measured at the beginning of first grade predicted the children's mathematical achievement scores as measured by a standardized tests one year later. Consistent with the other correlational studies discussed above, De Smedt et al. found that those children who exhibited a relatively larger numerical distance effect at the beginning of schooling had relatively lower math scores one year later. Put different, those children who had comparatively less precise representations of numerical magnitude at the beginning of formal schooling were found to have lower mathematical achievement scores at the end of first grade compared to their peers who started out with a comparatively small numerical distance effect.

Taken together, there is now a growing body of studies that suggest that individual differences in measures of numerical magnitude discrimination are related to measures of children's school-relevant mathematical achievement. These findings thereby provide a link between the basic systems of numerical magnitude representations, discussed above, and the development of higher-level numerical and mathematical skills.

Beyond discrimination tasks, it has also been shown that measures of children's number line estimation abilities are related to their scores on tests of mathematical achievement (Booth & Siegler, 2008; Siegler & Booth, 2004a). Specifically, both the accuracy and the degree of linearity of children's number line estimates have been found to be correlated with and predictive of their mathematical competence. As discussed above, the number line estimates of relatively older children can be modeled with a linear function, while estimation in kindergartners is best described by a logarithmic function. Thus the fact that the linearity of children's number line estimates is related to their math achievement suggests that those children with more mature and precise representations of numerical magnitude are those that have relatively higher mathematical achievement scores.

One may ask what underlies the relationship between basic numerical magnitude processing and higher level numerical and mathematical skills? It is becoming increasingly clear that numerical magnitude representations are fundamental to a large number of mathematical skills, including computing fractions (Siegler, Thompson, & Schneider, 2011). If one considers the case of arithmetic, it is clear that in order to solve calculation problems one needs to activate numerical magnitude representations. Before children can solve arithmetic problems by retrieving the solutions as facts they have to activate the numerical magnitudes associated with, for example, addends and then engaged in them in a computational transformation (in this case addition). Thus, the more precise the representations of numerical magnitude are the faster they can be accessed and used in the context of calculation.

Implications for educational measurement

The potential benefits of using measures of numerical magnitude as assessment tools

The above literature review suggests that individual differences in children's performance on measures of numerical magnitude processing such as symbolic and non-symbolic numerical magnitude comparison and number line estimation are related to variability in children's performance on standardized tests of mathematical achievement. In other words, measures that are thought to capture individual differences in the fundamental representation and processing of numerical magnitude are related to school-relevant measures of mathematical achievement. This raises the possibility that tasks such as number comparison and number line estimation can be used as educational measurements to characterize both large cohorts of children as well as provide a deeper understanding of the performance of individual children.

One of the exciting aspects of using measures such as number comparison and number line estimation to characterize individual differences in children's numerical and mathematical abilities is that these test can be administered long before children enter the formal classroom and can thus be used to assess the degree to which children are prepared for the demands that will be placed on them in the elementary school mathematics classroom. This might be especially important to know in low-income countries where little is known about the mathematical knowledge levels that children bring to school and the classroom instruction may be at level that is far above what children can be expected to learn given their prior knowledge. So these tests can be very useful as tools for the characterization of student's level of preparedness, which will in turn inform the kinds of information that is being taught in school.

In addition, tests of basic numerical magnitude processing could provide universal measures of numerical magnitude processing. As discussed above, humans share with animals a basic system for the representation of numerical magnitude that can be measured from infancy onwards. Thus it is likely that children across different cultures are born with similar neural mechanisms to construct representations of numerical magnitude and map them onto cultural symbols. In other words, the foundations of numerical magnitude processing are likely to be culturally invariant and interact with cultural factors over the course of development and learning. Thus tests of basic (especially non-symbolic) numerical magnitude processing may be especially well suited for assessing children's numerical processing competencies across cultures and educational systems.

Furthermore such tests can be used to identify children who might be at risk of developing mathematical difficulties, such as developmental dyscalculia. By using non-symbolic items (such as dot arrays) it is possible to measure individual difference in numerical magnitude processing even before children have an understanding of the meaning of numerical symbols (such as Arabic numerals or number words). From the reading literature it is clear that individual differences in basic, early-developing competencies, such as phonological awareness are critical for reading development (Goswami, 2003; Guttorm et al., 2005; Stanovich, 1986). Can we get to a similar level of early assessment in the domain of numerical and mathematical abilities? With the scientific insights into the building blocks of numerical and mathematical abilities discussed above, it is possible that individual difference in fundamental numerical magnitude processing abilities can be characterized early and provide educators with tools to assess both children at risk of falling behind as well as gaining a better understanding of the level of understanding bring to the formal educational environment.

Current standardized tests of numerical and mathematical processing do not routinely contain items that tap into the basic representations of numerical magnitude. Most of the available standardized tests consist of either calculation or so-called 'Mathematical Reasoning' tests. For the calculation tests, children are presented with a series of calculation problems of all four arithmetic operations and are required to solve these problems under timed or untimed conditions. For the tests of 'Mathematical Reasoning' children are asked to solve diverse items such as judging the order of items of varying sizes or solving illustrated word problems. What is currently lacking from educational assessments of children's numerical and mathematical skills are 'processing measures'. In other words, measures that capture something fundamental about the very basic representations that children require in order to process numerical information in the context of calculation and 'Mathematics Reasoning'. It is surprising, given the literature reviewed above, that such measures are not currently available on the educational testing/assessment market. In general, it is not always clear exactly what empirical evidence is drawn upon to construct assessments of numerical and mathematical abilities. For example, most tests of 'Mathematical Reasoning' contain very heterogeneous sets of items and it is not clear what central construct is being tested here.

As a contrast consider currently available reading assessments. In addition to tests of children's ability to read words and sentences (either silently or aloud), there exist many assessments to tap into children's basic understanding of the relationship between letters and speech sounds (phonics) as well as their ability to understand that words consist of different sounds (phonological awareness). Since it has become well established that phonological awareness is a fundamental building block of reading (Stanovich, 1986), reading assessments typically measures both reading itself as well as processing measures of phonics, phonological awareness and other measures or processes considered to be fundamental to enabling children to become fluent readers. In view of the literature reviewed above, the same maybe possible for assessments of children's numerical and mathematical competencies by combining traditional tests of calculation with processing measures such as number comparison and number line estimation that may yield a fuller profile of children's level of ability.

It is important to note in this context that any such processing measures should be used in addition to traditional calculation tests and should, in my view, not replace these tests and can, as discussed above, be used before children are formally introduced to calculation.

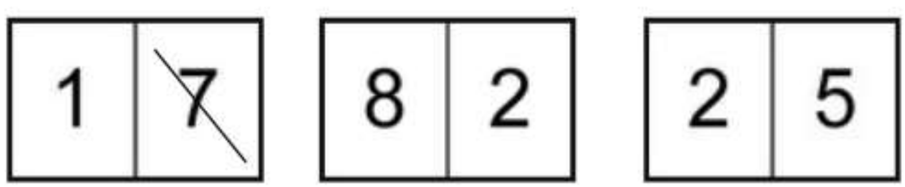
Design considerations

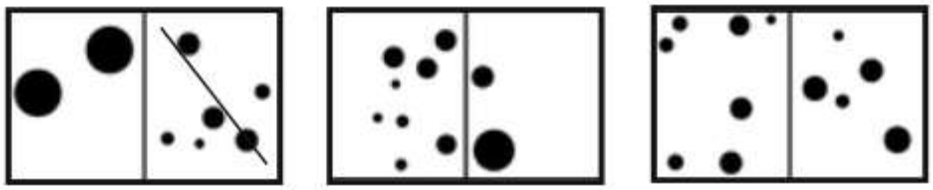
Given that standardized measures of numerical magnitude processing are currently, to the best of my knowledge, not available, there are a number of design issues that need to be taken into consideration. One important design choice is whether to use computers or pencil and paper as the response modalities. One clear advantage of computerized measurement is that reaction time data can be acquired. Many of the above studies in which numerical magnitude processing has been shown to be associated with and predictive of individual differences in children's mathematical achievement levels used reaction time measures of variables such as the distance and ratio effects. However, computers are expensive and require specialized software. Furthermore, when working with individuals who may have only had limited exposure to computers, the use of computers to administer test may lead to difficulties which may preclude accurate measurement of children's competencies.

While pencil and paper tests do not allow the measurement of item by item reaction times, they can nevertheless be used to measure how many items are completed within a given time period. (Durand, Hulme, Larkin, & Snowling, 2005) employed this kind of items/time design to assess children's number comparison abilities and relate their performance on symbolic number comparison to their scores on a test of mathematical achievement. In their design children were shown pairs of Arabic digits and were asked to use a pencil to strike through or cancel out the numerically larger of the large digits. The test lasted for 30 seconds. The authors found that the number of items (number comparisons) children could complete in the 30 seconds time period was correlated with their mathematical, but not their reading, scores. These data illustrate that a simple paper and pencil version of the number comparison tasks can be used to capture meaningful individual differences in children's mathematical achievement.

My graduate student, Nadia Nosworthy, and I, are currently testing a similar extended test. Specifically, children are asked to perform the cancellation task with both symbolic and non-symbolic stimuli (see Figure 10 below) and we have manipulated the numerical ratio such that later items in the test have increasingly larger numerical ratios, resulting in more difficulties in discriminating between them.

Figure 10. *Example stimuli from paper-and-pencil numerical magnitude comparison tasks for both symbolic and non-symbolic stimuli. Children have to cross out the larger of the two numbers*





The test consists of 56 symbolic comparison trials (Arabic numerals) and 56 non-symbolic comparison (arrays of dots) trials. The symbolic and non-symbolic trials are presented in blocks, such that children either perform all the symbolic and non-symbolic comparisons separately. In other words, there is no mixing of symbolic and non-symbolic formats in this test. This is to avoid children having to switch between formats and to get separate measures of children's ability to compare symbolic and non-symbolic numerical magnitudes. Furthermore, this also allows for the use of only the non-symbolic items if children cannot process Arabic numerals (particularly preschool children). This might be particularly important when testing illiterate populations as they may not have experience with symbolic representations of number but can nevertheless compare non-symbolic numerical magnitudes.
All the pairs consist of combinations of number 1-9. The items are presented as shown in Figure 10 above on paper and children use a pencil to cross out the number symbol or dot array that they judge to be numerically larger and then move onto the next item. Before the test starts children are given several practice items. The test is timed using a conventional stopwatch and children are given 1 minute to complete either the symbolic or non-symbolic items. Children are then given a break and move onto either the non-symbolic or symbolic items (depending on which they completed first), for which they are again given 1 minute to complete as many of the 56 items as they can. Children are told that they should try to solve as many items as possible in the time allocated and do so as accurately as they can.

So far this test has been administered to children individually with an experimenter giving the instruction and operating the Stopwatch. However, this test could very easily be administered in group settings, such as large classrooms. It would require that the record books be distributed to children and the instructions to be clearly delivered to the whole group. The operation of the stopwatch could be undertaken by a teacher or teacher assistant. Alternatively, if a stopwatch is not available then the teacher can use a wall clock or personal watch and time the test approximately. It is, however, critical that this test be timed. If the test is untimed then children may be able to complete all items but this would not test the efficiency with which they are able to process numerical magnitude. The test is very simply and requires minimal instruction. The test contains practice items, which children should complete first and teachers should make sure that all children in the classroom have understood the instructions by checking their performance on the practice items. If a child has erroneous responses on all the practice items, then the teacher needs to go over these items again with the child to make sure they understand the tasks that they are being asked to complete. Once all children have understood the tasks, the testing can commence. It is important that all children stop working on the test when the pre-specified time is up. Therefore teachers and teaching assistants need to make sure that all pencils are put down at roughly the same time.

The test described above is currently being evaluated in a sample of around 350 students in London, Ontario Canada and preliminary results indicate strong correlations between children's accuracy at performing this test and individual differences in math achievement. Furthermore, by using multiple regression analyses we have been able to establish that individual difference in children's performance on this simple magnitude comparison measure explain significant variance in children's mathematics achievement over and above the variance explained by measures of verbal and spatial working memory as well as phonological processing. The results therefore indicate that this simple test can capture individual differences in performance that are not explained by other factors, such as working memory, which have previously been associated with individual differences in mathematics performance. Following this pilot phase, alterations will be made to improve the test. The test will then go to full age-based and grade-based standardization, so that age and grade norms can be established. This test is designed for the assessment of children in K-3, since children after 3rd grade are likely to perform at near perfect levels (ceiling) on such a test. Of course, it is possible to make the test more difficult by going from single to double digits and increasing the ratios of the numbers to be compared. Furthermore, the time in which children are asked to complete the test can also be altered.

The importance of timing

There are two issues that are important to carefully consider when designing measures of basic symbolic and non-symbolic numerical magnitude processing. The first one is timing. In the domain of reading it is well recognized that fluency is an important predictor of literacy development (Ehri, 1995) and tests often use number of words or letters read in a limited time window in assessments (Woodcock, McGrew, & Mather, 2001). The same is true of mathematics, where the efficient ability to access numerical magnitude information from numerical symbols is likely to aid calculation efficiency (Holloway & Ansari, 2009). Thus tests that assess numerical magnitude processing in children should test these abilities under timed conditions. When symbolic representations of numerical magnitude (such as Arabic numerals) are used, timed tests tap into the ability of children to quickly access the semantic meaning (numerical magnitude) that the symbols represent and then to perform a comparison of the semantic representation of the two numerals. In non-symbolic numerical magnitude comparison, a timed test assesses the efficiency with which children can deploy their approximate number system to generate approximate number representations and compare them. If such tests are administered in an untimed fashion they run the risk of introducing large strategic variability between children (some children may use counting to compare non-symbolic numerical magnitudes) and ceiling effects (given that these tests, especially those using single digits, are fairly simple, given time, most children may be able to complete them). Thus it is important to pay attention to number of items correctly/time rather than number of items correctly alone. This is particularly true for the number comparison or dot estimation type of items.

Ideally, reaction times can also be used to construct within-task measures of individual differences such as individual differences in the numerical distance effect. In other words, instead of simply using number of items correctly solved/time one could compare number of items correctly solved/time for small and large distances separately. If computer technology is available the reaction times for each comparison should be measured to be able to quantify the effect of reaction time for each individual participant.

Having made the point that timing the test administration is important, this may not be true for all items on a test. For example, number line estimation does not require timing, since the principle variable is the difference between the veridical position of a numerical magnitude on the number line and the position indicated by the participant. In all the research on number line estimation, its development and relationship to arithmetic achievement, untimed measures have been used. Furthermore, if counting is assessed children should not be speeded, as otherwise they resort to estimation strategies.

Thus, while for magnitude comparison tasks it is important to use timed measures, there are other aspects of basic numerical magnitude processing that might be included in a test that do not require a speeded response, such as dot counting. In a dot counting task, children are presented with an array of dots are asked to count the array to determine exactly how many dots are displayed.

Another important consideration when designing tests that required speeded/timed responses is to acknowledge that reaction times vary dramatically across chronological age and differ between literate and illiterate populations (Abadzi, 2006) and thus careful piloting needs to be undertaken to decide what the optimal time limit is in any particular population to avoid ceiling and floor effects.

The importance of choosing items

Of key importance when designing an assessment of early numerical abilities is to carefully choose items and to ensure that enough items are presented to tap into particular competencies in a valid and reliable fashion. It is key that tests of early mathematics include non-symbolic items that tap into children's ability to match and compare non-symbolic numerical magnitude. This is important for the following reasons: a.) as discussed above, non-symbolic representations are foundational and can be found even in young infants and non-human primates. It is therefore to ascertain that children have these foundational competencies. Furthermore b.) when working with children who are either illiterate or have little access to formal schooling, non-symbolic representations of numerical magnitude can help to assess their level of understanding of numerical magnitude independent of their knowledge of cultural, symbolic representations.

While tests, such as the Early Grade Math Assessment (EGMA) developed by the Research Triangle Institute do contain non-symbolic items, this is only one small part of the test and non-symbolic items are used in the context of exact counting, rather than estimation or comparison. Therefore this test relies very heavily on symbolic number processing skills to assess children's number processing abilities. A combination of symbolic and non-symbolic items may improve the strength with which this measure will be able to detect individual differences. Moreover, asking children to match symbolic and non-symbolic representation of number, such as deciding whether a dot array and Arabic numerals represent the same numerical magnitude, can be used to assess the degree to which children are able to map between symbolic and non-symbolic representations. As discussed above, one key foundational competency in number processing is learning the meaning of symbolic representations of numerical magnitude, such as number words and digits. This requires the mapping between symbolic and non-symbolic representations of numerical magnitude. In fairness to the EGMA test, it is a test that is designed for 'Early Grade Assessment', and thus the focus is on the kinds of numerical skills that children are expected to acquire during the early years of formal schooling. However, the test may be even stronger if it considered number processing skills that children exhibit even before they learn to use symbols to represent and process numerical magnitudes and thereby tested for the foundations. In this context it is very important to point out that the suggestion is not that non-symbolic items dominate early assessments, but instead that they be added and become one component of such measures and that tests also measure how well children can match symbolic with non-symbolic representations (a test of the strength of the understanding of the meaning of number symbols).

Another important aspect to consider is how the difficulty of items is manipulated. In most standardized tests difficulty increases as a function of the test trial, such that simpler items are presented before more difficult ones. This is, for example, done very well in the EGMA test where single digit numeral knowledge is tested before children's ability to use these symbols to calculate.

The EGMA test is structured around two constructs: 1. Number and Operations and 2. Geometry. The strength of the test is that it assesses a host of skills, including Oral Counting Fluency, One-to-One Correspondence, Number Naming Fluency, Quantity Discrimination, Number Line Estimation, Word Problems, Addition and Subtraction, Shape Recognition, Shape Attributes and Pattern/Number Extension. Another strength of the EGMA test is that it can be administered in around 15 minutes. However, as discussed above, the test would benefit from additional non-symbolic items. Furthermore, there is a need to have a sufficient number of items to be able to assess children's ability using measures that go beyond mean accuracy. Many of the EGMA subtests contain very few items, which may preclude detailed assessments of the processes underlying children's performance on the measures administered. For example in number line estimation enough items need to be presented in order to establish whether a child's mapping of number onto space (the number line) can be best characterized by a linear or a logarithmic function (for a review of this important difference, see above). Similarly to estimate distance and ratio effects in the context of number comparison (for a review of these effects see above), items need to be carefully selected to span a range of distances/ratios and multiple trials per ratio need to be administered to get a reliable assessment of the effects.

Many of the above points also apply to another assessment tool used in low-income countries: the Math Test constructed by the 'Assessment Survey Evaluation Research Centre' (ASER Centre: www.asercentre.org). This brief test assesses number recognition, subtraction and division. All items are symbolic and it is not clear from the test why particular these particular competencies are being assessed. Furthermore, being able to recognize a numeral is not the same as being able to use the numerical magnitude information represented by that numeral. Therefore, in addition to testing recognition and/or number reading, number comparison tasks should be included to assess the degree to which children can use the numerical magnitudes represented by the numerals in the context of a comparison task. The test does not contain any measures of basic numerical magnitude processing (using either symbolic or non-symbolic items).

A final point concerns the importance of choosing items on the basis of peer-reviewed evidence. Only items should be included that have been found to be reliable correlates or predictors of individual differences in children's numerical and mathematical understanding should be used. It is therefore critical that test construction involve consultation by developmental cognitive scientists who specialize in the development of children's number processing. Such interactions will ensure that items are chosen on the basis of the latest peer-reviewed evidence, that such items are age appropriate and that sufficient items are selected for different subtests to allow for the rich characterization of children's numerical and mathematical abilities. Tests should not be based on intuitions or on what is being administered as part of a particular curriculum but the items on a test should be motivated by what is known about children's numerical competencies, what the key milestones in this learning process are and what tasks best describe children's foundational competencies.

Implications for educational practice

One final question that the present review will consider concerns the implications of the behavioral and neuroscientific research on basic numerical magnitude representation for educational practice. I should point out that I am not an expert in pedagogy and thus cannot propose specific activities but merely provide pointers towards certain aspects that, given the literature discussed above, could inform educational practice in preschool and early elementary school.

First of all, I strongly believe that sharing the above-reviewed research with teachers will lead to changes in their practice and that teachers themselves will be the best 'translators' of the empirical evidence surrounding the foundations of numerical and mathematical skill development (Ansari & Coch, 2006). Many teachers may not know that even very young infants can process non-symbolic numerical magnitudes and that we share representations for the processing of numerical magnitude with non-human animals. Such knowledge will allow teachers to reappraise the abilities of the children in their classrooms and think about ways in which to harness the skills that students come to their classrooms.

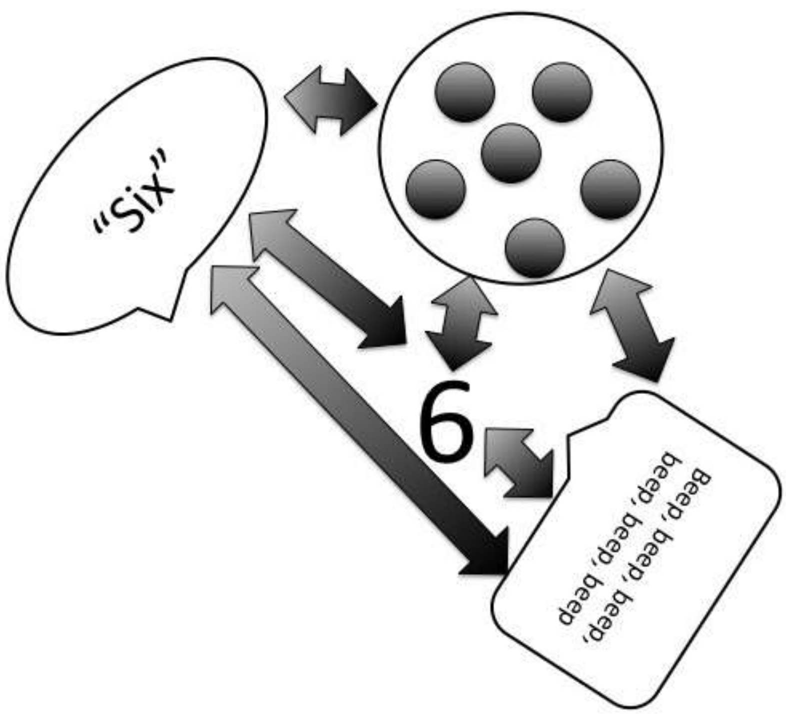
More directly related to practice, the literature above suggests that early education in numerical and mathematical processing should focus on strengthening children's ability to process numerical magnitudes through activities that might involve estimation, comparison both with symbolic and non-symbolic representations of numerical magnitude. This can be achieved through games that involve matching symbolic with non-symbolic numerical magnitudes. For example, one can make cards that have different numbers or arrays of dots printed on the back. Using the same format as memory games with cards, all the cards are put face down and the child has to turn cards over and match them on the basis of their numerical magnitude (3 dots matched with the numeral 3). Such a game requires enumeration through counting and comparison both within and across symbolic and non-symbolic formats. Other games may involve ordering the same cards in ascending and descending order and creating one ordered with only symbolic and another with non-symbolic items. In addition to learning the meaning of individual numerical symbols, children also need to acquire an understanding of the number sequence (order) and the relationships (larger than, smaller than) of numerical magnitudes within this sequence. Other games can use the number line and objects moving positions down and up the number line as a function of rolling a dice. There are many simple activities that can be used to help children strengthen their understanding of numerical magnitude, the relationship between numerical magnitude and the meaning of numerical symbols. This is not to say that early education should revolve around training children on numerical magnitude comparison and estimation tasks. In fact it is important to distinguish between an experimental measure of basic numerical magnitude processing and methods for teaching children about numerical magnitudes. Simply repeating items used in the context of an assessment tool will probably not yield significant learning.

Educators need to devote time to strengthening children's understanding of numerical magnitudes and the relationships between them. Recent research has shown that playing linear board games, such as Snakes and Ladders, leads to improvements in children's numerical magnitude representations (Siegler & Booth, 2005). During those games children have to relate magnitudes to one another and move up and down the number line. Such activities are affectively engaging for children and thus strengthening of numerical magnitude processing skills are a by-product of game-like activities. Such games will invariably involve culturally-appropriate materials.

One of the key challenges in learning about numerical magnitudes is developing understanding that numerical magnitudes are abstract and that different external formats of presentation can all refer to the same internally represented numerical magnitude. This importance of linking across different formats of numerical magnitude representations is illustrated in Figure 11.

Thus part of early math education should focus on helping children to integrate across various formats for the representation and processing of numerical magnitude to build an increasingly abstract representation of numerical magnitude. In this context educational activities that help children understand that symbolic stimuli such as digits refer to numerical magnitudes that can be represented non-symbolically (with sets of objects) is important. From a curriculum perspective it is important to spend time on helping children integrate different representations of numerical magnitude. Thus activities that involve moving between different symbolic and non-symbolic representations. For example, activities should be designed that help children to develop the ability to flexibly translate between different ways of representing the same numerical magnitude. This is a central component of the highly successful 'Number Worlds' program (Griffin, 2004). This could be achieve through educational activities in which children are asked to match different external ways of representing numerical magnitudes with one another (such as matching digits with dots, or linearly vs. randomly arranged objects on the basis of number). Another way is to ask children to identify the number that does not match with any of the others. Specifically, in such an exercise children are presented with say 7 dots, the numeral 7, 7 marks on a page, 7 pictures of fruit and the numeral 10 and they have to identify that the numeral 10 is the odd one out (since it differs from the other stimuli in numerical magnitude). For some excellent examples of exercises that strengthen children's ability to translate between and relate to one another different representations of number, please see the primary school booklets of the JUMP Math Program (www.jumpmath.org)

Figure 11. *Illustrating the abstract nature of numerical magnitude representations. The same numerical magnitude can be represented in multiple external formats (both symbolic and non-symbolic) across different modalities.*



Summary and Conclusions

The present paper provides an overview of the foundational systems of numerical magnitude representation that form the basis of numerical and mathematical abilities. The paper reviews both behavioral and neuroscientific data from non-human animals, infants, children and adults. These data reveal a system for the approximate representation of numerical magnitude that can be reliably measured with experimental paradigms such as numerical magnitude comparison and estimation and is associated with the activation of regions of the parietal cortex in the brain. Effects such as the numerical distance and ratio effect can be used to characterize individual difference in numerical magnitude processing. Moreover, while the basic signatures of the system for the representation and processing of numerical magnitude appear qualitatively similar across species and development, the precision (or acuity) of the system increases over developmental time, allowing children to discriminate between increasingly finer differences in numerical magnitude. Thus the representations of numerical magnitude become less approximate over the course of learning and development.

Importantly from an educational perspective, the above review of the literature reveals that individual differences in numerical magnitude representations and particularly the fluency of processing symbolic representations of numerical magnitude are related to and predictive of children's school-relevant mathematical achievement (as measured by their performance on standardized tests of numerical and mathematical skills). In other words, the empirical evidence reviewed above provides a link between, on the one hand, foundational systems for the representation and processing of numerical magnitude and, on the other, individual differences in children's mathematical achievement levels.

These findings opens up the possibility of using indices of numerical magnitude processing skills as measures of children's numerical abilities across the globe. By doing so, such measures could serve as indicators of the level of foundational skills exhibited by children who are going to be entering formal education or are within the first few years of their formal education. Such assessment tools would mirror the popular processing measures used in reading assessments, such as tests of children's phonological awareness skills. Moreover, such tools for assessment have the potential to inform the design preschool programs and help to screen for children who are at risk of falling behind in formal mathematics education. Since numerical magnitude representations can be measured using non-symbolic stimuli (such as arrays of dot), such measured could be administered long before children understand the meaning of abstract, cultural symbols for numerical magnitude, such as Arabic numerals. Furthermore, the literature reviewed above also has direct implication for early mathematics instruction by highlighting the importance of fostering children's numerical magnitude representations and helping children built abstract representations of numerical magnitude.

In closing, it should be noted that mathematical development is multifaceted and involves the complex interactions between multiple factors, such as working memory, strategy use, attention and reasoning capacities. The claim of this review is therefore not that the processing and representation of numerical magnitude is the only important variable in children's number development. Rather the argument is that numerical magnitude processing provides the universal foundations upon which higher-level numerical and mathematical skills are built and thus represent a target for early assessment and education.

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