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APPLICATIONS OF **LORENZ** CURVES  
IN ECONOMIC ANALYSIS

by

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**NOTE:** Discussion Papers are preliminary materials circulated to stimulate discussion and critical comment. References in publication to Discussion Papers should be cleared with the author(s) to protect the tentative character of these papers

## I. INTRODUCTION

The Lorenz curve relates the **cumulative** proportion of **income units** to the **cumulative** proportion of income received when units are arranged in ascending order of their income. In the past the curve has been mainly used as a convenient graphical device to represent the size distribution of **income** and wealth.

The interest **in** the Lorenz curve technique has been recently revived by **Atkinson** [ 1 ] who provided a theorem relating the **social** welfare function and the Lorenz curve. **He** showed that the ranking of income distributions according to the Lorenz curve criterion is identical with the ranking implied by aggregate economic welfare regardless of the form of the welfare function of the individuals (except that it be increasing and concave) provided the Lorenz curves do not intersect. However, if the Lorenz curve do intersect, one can always find two functions that will rank them differently. Das Gupta, Sen and Starrett [ 2 ] have **shown** that this result is **infact** more general and does not depend on the assumption that the welfare functions should necessarily be **additive**.

In the present paper the Lorenz **curve** technique is used as a **tool** to introduce distributional considerations in economic analysis. The **concept** of Lorenz curve has been extended and generalized to study the **relationships** among the distributions of different economic variables. The **generalized Lorenz** curves are called concentration curves and the Lorenz **curve** is only a special case of such curves, **viz.,** the concentration curve for income.<sup>1/</sup>

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<sup>1/</sup> Professor Mahalanobis [ 6 ] used **concentration** curves to describe the consumption pattern for different **commodities** based on the National sample Survey Data. See also Roy, Chakravarty and Laha [ 7 ]

Section 2 gives the derivation of the Lorenz curve. Some **theorems** relative to the concentration curve of a function and its elasticity are provided in Section 3. These theorems provide the basis to study relationships **among** the distributions of different economic variables. Applications of the theorem are **discussed** in Section 4.

## 2. THE WRENZ CURVE

Suppose that income  $X$  of a family is a random variable with probability density **function**  $f(X)$ . Then the distribution function  $F(x)$  is defined as:

$$F(x) = \int_0^x f(X) dX \quad (2.1)$$

and this function can be interpreted **as** the proportion of families having income **less than** or equal to  $x$ .

If it is **assumed** that the mean  $E(X) = \mu$  of the **distribution** exists and  $X \geq 0$ , then the **first moment** distribution function of  $X$  is defined as:

$$F_1(x) = \frac{1}{\mu} \int_0^x Xf(X) dX \quad (2.2)$$

The Lorenz curve is the relationship **between**  $F(x)$  and  $F_1(x)$ . The **graph** of the curve is represented in a unit square. The **equation** of the line  $F_1 = F$  is called the egalitarian line and if the Lorenz curve coincides with this line it implies that each family **receives** the **same** income.

The most widely used measure of inequality is Gini's Index which is equal to twice the area between the Lorenz curve and the egalitarian line. It can be written as:

$$G = 1 - 2 \int_0^{\infty} F_1(x) f(x) dx \quad (2.3)$$

and it varies from zero to one.

### 3. THE CONCENTRATION CURVES

Let  $g(X)$  be a continuous function of  $X$  such that its first derivative exists and  $g(X) \geq 0$  for  $X \geq 0$ . If  $E[g(X)]$  exists, then one can define:

$$F_1[g(x)] = \frac{1}{E[g(X)]} \int_0^x g(X) f(X) dx \quad (3.1)$$

where:

$$E[g(X)] = \int_0^{\infty} g(X) f(X) dx \quad (3.2)$$

so that  $F_1[g(x)]$  is monotonic increasing and  $F_1[g(0)] = 0$  and  $F_1[g(\infty)] = 1$ .

The relationship between  $F_1[g(x)]$  and  $F(x)$  will be called the concentration curve of the function  $g(x)$ .

It can be seen that the Lorenz curve of income  $x$  is a special case of the concentration curve for the function  $g(x)$  when  $g(x) = x$ .

The above generalization of the Lorenz curve was suggested by Professor P.C. Mahalanobis to describe the consumer behaviour pattern with

respect to different commodities.

The relationship between  $F_1 [g(x)]$  and  $F_1(x)$  will be called the relative concentration curve of  $g(x)$  with respect to  $x$ . Similarly, let  $g^*(x)$  be another continuous function of  $x$ , then the graph of  $F_1 [g(x)]$  vs  $F_1 [g^*(x)]$  will be called the relative concentration curve of  $g(x)$  with respect to  $g^*(x)$ . Let  $\eta_g(x)$  be the elasticity of  $g(x)$  with respect to  $x$ , then:

$$\eta_g(x) = \frac{g'(x)}{g(x)} x \quad (3.3)$$

where  $g'(x)$  is the first derivative of  $g(x)$ .

Similarly denote  $\eta_{g^*}(x)$  as the elasticity of  $g^*(x)$  with respect to  $x$ .

We can now state the following theorem:

**THEOREM 1:** The concentration curve for the function  $g(x)$  will lie above (below) the concentration curve for the function  $g^*(x)$  if  $\eta_g(x)$  is less (greater) than  $\eta_{g^*}(x)$  for all  $x \geq 0$ .

Proof of the Theorem 1

Using the equation (3.1) we obtain:

$$\frac{d F_1 [g(x)]}{dx} = \frac{g(x) f(x)}{E [g(x)]} \quad (3.4)$$

and

$$\frac{d F_1 [g^*(x)]}{dx} = \frac{g^*(x) f(x)}{E [g^*(x)]} \quad (3.5)$$

which give the slope of the relative concentration curve of  $g(x)$  with respect to  $g^*(x)$  as:

$$\frac{dF_1 [g(x)]}{dF_1 [g^*(x)]} = \frac{E [g^*(x)]}{E [g(x)]} \times \frac{g(x)}{g^*(x)} \quad (3.6)$$

The equation (3.6) implies that the relative concentration curve is monotonic increasing. Since the curve must pass through (0,0) and (1,1) it follows that a sufficient condition for  $F_1 [g(x)]$  to be greater (less) than  $F_1 [g^*(x)]$  is that the curve be convex (concave) from above. To establish curvature we obtain the second derivative of  $F_1 [g(x)]$  with respect to  $F_1 [g^*(x)]$  as:

$$\frac{d^2 F_1 [g(x)]}{dF_1^2 [g^*(x)]} = \frac{(E [g^*(x)])^2}{E [g(x)]} \times \frac{g(x)}{g^{*2}(x)} \times \frac{[\eta_g - \eta_{g^*}]}{x f(x)} \quad (3.7)$$

the sign of the second derivative is given by the sign of  $\eta_g(x) - \eta_{g^*}(x)$ . Thus the second derivative is positive (negative) if  $\eta_g$  is greater (less) than  $\eta_{g^*}$  for all  $x$ . Hence the concentration curve for  $g(x)$  is above (below) the concentration curve for  $g^*(x)$  if  $\eta_g(x)$  is less (greater) than  $\eta_{g^*}(x)$  for all  $x > 0$ .

#### Special Cases

Let  $g^*(x) = \text{constant}$  for all  $x > 0$ , then the elasticity  $\eta_{g^*}(x) = 0$  and  $F_1 [g^*(x)] = F(x)$  which is the equation of the egalitarian line. Thus we have the following corollary.

COROLLARY 1: The concentration curve for the function  $g(x)$  will be above (below) the egalitarian line if  $\eta_g(x)$  is less (greater) than zero.

The proof of Corollary 1 is also given by Roy, Chakravarti and Laha [ 7 ]. Next we assume that  $g^*(x) = x$  so that  $\eta_{g^*}(x) = 1$  and the concentration curve for  $g(x)$  is now the Lorenz curve for the distribution  $x$ . It follows from the Corollary 1 that the Lorenz curve for  $x$  lies below the egalitarian line and therefore the curve is concave from above. Further, from Theorem 1 we have the following Corollary.

COROLLARY 2: The concentration curve for the function  $g(x)$  lies above (below) the Lorenz curve for the distribution of  $x$  if  $\eta_g(x)$  is less (greater) than unity for all  $x \geq 0$ .

If the function  $g(x)$  has the unit elasticity for all  $x \geq 0$ , the second derivative for the relation concentration of  $g(x)$  with respect to  $x$  will be zero which implies that slope of the relative concentration curve will be constant for all values of  $x$ . Since the curve must pass through (0,0) and (1,1) it means that the relative concentration of  $g(x)$  with respect to  $x$ , coincides with the line, (0,0) and (1,1). Hence  $F_1[g(x)] = F_1(x)$  for all  $x$ ; which proves the following:

COROLLARY 3: The concentration curve for  $g(x)$  coincides with the Lorenz curve for  $x$  if  $\eta_g(x) = 1$  for all values of  $x$ .

It should be pointed out that the concentration for  $g(x)$  is not the same thing as the Lorenz curve for  $g(x)$ . We will now discuss the condition under which both are identical.

Let  $y = g(x)$  be a random variable with probability density function  $f^*(y)$  and the distribution function  $F(y)$ , and if mean of  $y$  exists, the first moment distribution function of  $y$  is given by:

$$F_1^*(y) = \frac{1}{E(y)} \int_0^y y f^*(y) dy, \quad (3.8)$$

then  $(F^*(y), F_1^*(y))$  is a point on the Lorenz curve for  $g(x)$ . The following theorem gives the conditions under which:

$$F^*(y) = F(x) \text{ and } F_1^*(y) = F_1[g(x)] \quad (3.9)$$

for all values of  $x$ .

**THEOREM 2:** If  $g(x)$  is strictly monotonic and has a continuous derivative  $g'(x) > 0$  for all  $x$ , then the concentration curve for  $g(x)$  coincides with the Lorenz curve for the distribution of  $g(x)$ .

Proof of the Theorem 2

Under the assumption that  $g(x)$  is strictly monotonic and has a continuous non-vanishing derivative in the region  $x \geq 0$ , the probability density function of  $y$  is given by

$$f^*(y) = f[h(y)] |h'(y)| \quad (3.10)$$

where  $x = h(y)$  is the solution of  $y = g(x)$ .

Let us now consider the graph of  $F(x)$  vs  $F^*[g(x)]$  which has the slope

$$\frac{dF^*[g(x)]}{dF(x)} = \frac{f^*(y)}{f(x) h'(y)} \quad (3.11)$$



which on using (3.10) becomes one if  $h'(y) > 0$ .  $h'(y)$  is obviously greater than zero for all  $y$ . Further since  $g'(x) > 0$  and the curve must pass through  $(0,0)$  and  $(1,1)$  it implies that the curve  $F [ g(x) ]$  vs  $F(x)$  which has constant slope one must coincide with the line passing through  $(0,0)$  and  $(1,1)$ . Hence  $F [ g(x) ] = F(x)$ .

Similarly it can be proved that the graph of  $F_1 [ g(x) ]$  vs  $F_1 [ g(x) ]$  has slope one if  $h'(y) > 0$ . Since the curve passes through  $(0,0)$  and  $(1,1)$ , it must coincide with the straight line joining  $(0,0)$  and  $(1,1)$  which implies  $F_1 [ g(x) ] = F_1 [ g(x) ]$ . This proves the theorem.

DEFINITION 1: The function  $g(x)$  is said to be Lorenz superior (inferior) to another function  $g^*(x)$  if the Lorenz curve for  $g(x)$  lies above (below) the Lorenz curve for  $g^*(x)$  for all  $x \geq 0$ .

It follows from the definition of Gini-Index that the distribution generated from function  $g(x)$  will have lower (higher) value of Gini-Index than the distribution generated from  $g^*(x)$  if  $g(x)$  is Lorenz superior (inferior) to  $g^*(x)$ .

If the functions  $g(x)$  and  $g^*(x)$  are strictly monotonic and have continuous derivatives strictly greater than zero, then from Theorem 2 it follows that their concentration curves coincide with their respective Lorenz curves. Then using Theorem 1 we obtain the following Corollary.

COROLLARY 5: If the functions  $g(x)$  and  $g^*(x)$  are strictly monotonic and have continuous derivatives strictly greater than zero,

then  $g(x)$  is Lorenz superior (inferior) to  $g^*(x)$   
if  $\eta_g(x)$  is less (greater) than  $\eta_{g^*}(x)$  for all  
 $x \geq 0$  .

Again if we put  $g^*(x) = x$  so that  $\eta_{g^*}(x) = 1$  then Corollary 5 leads to the following Corollary.

COROLLARY 6: If  $g(x)$  is strictly monotonic and has a continuous deriva-  
tive  $g'(x) > 0$  for all  $x$ , then  $g(x)$  is Lorenz superior  
(inferior) to  $x$  if  $\eta_g(x)$  is less (greater than  $\eta_{g^*}(x)$   
for all  $x \geq 0$  .

DEFINITION 2: The concentration index for  $g(x)$  is defined as one minus  
twice the area under the concentration curve for  $g(x)$ .

In our notation, the concentration index for  $g(x)$  is given by:

$$C_g = 1 - 2 \int_0^{\infty} F_1 [g(x)] f(x) dx.$$

It is to be noted that if  $g(x) = \text{constant}$ , the concentration curve coincides with the egalitarian line so that  $C = 0$ . If  $g(x) = \alpha x$  where  $\alpha$  is any constant, then the concentration is equal to the Gini-Index of  $x$ . Further, if  $g(x) > 0$  for all  $x$ , then  $C_g$  is always positive and will be equal to the Gini-Index of the function  $g(x)$ . Finally if  $g(x) < 0$  for all  $x$ , then the concentration curve for  $g(x)$  is above the egalitarian line and  $C_g$  will be equal to minus times the Gini-Index of the function  $g(x)$ . The  $C_g$  lies between  $-G_g$  to  $+G_g$ , if  $G_g$  is the Gini-Index of the function  $g(x)$ .

THEOREM 3: If  $g(x) = \sum_{i=1}^k g_i(x)$  so that  $E[g(x)] = \sum_{i=1}^k E[g_i(x)]$

where  $E$  is the expected value operator, then:

$$E[g(x)] F_1[g(x)] = \sum_{i=1}^k E[g_i(x)] F_1[g_i(x)] \quad (3.13)$$

Proof of the Theorem 3

Substituting  $g(x) = \sum_{i=1}^k g_i(x)$  in (3.1) gives:<sup>3/</sup>

$$F_1[g(x)] = \frac{1}{E[g(x)]} \sum_{i=1}^k \int_0^{\infty} g_i(x) f(x) dx \quad (3.14)$$

Now  $F_1[g_i(x)]$  is given by:

$$F_1[g_i(x)] = \frac{1}{E[g_i(x)]} \int_0^x g_i(x) f(x) dx \quad (3.15)$$

Which on substituting in (2.13) gives the result stated in Theorem 3.

Let  $g(x) = a+bx$  so that  $E[g(x)] = a+bu$ , where  $E(x)=\mu$ ; then  $g(x)$  can be treated as the sum of two functions, viz,  $a$  and  $bx$ . Hence from Theorem 3 we obtain:

$$F_1[a+bx] = \frac{1}{(a+bu)} [a F(x) + bu F_1(x)] \quad (3.16)$$

Because the concentration curve for a constant function coincides with the egalitarian line. The equation (3.16) can also be written as:

$$F_1[a+bx - F_1(x)] = \frac{a}{a+bu} [F(x) - F_1(x)] \quad (3.17)$$

<sup>3/</sup> The interchange of summation sign and integral sign is permissible if  $k$  is finite.

Since  $F(x) \geq F_1(x)$  for all  $x$  it implies that the concentration curve for a linear function  $(a + bx)$  lies above (below) the Lorenz curve for  $x$  if  $a$  is greater (less) than zero. Further if  $b > 0$ , the function  $g(x) = a - bx$  is a monotonic increasing function of  $x$ , from Theorem 2 it follows that the concentration curve for  $(a + bx)$  coincides with the Lorenz curve of function  $(a + bx)$ . Thus we have the following corollary.

COROLLARY 7: If  $b > 0$ , then the linear function  $(a + bx)$  is Lorenz superior (inferior) to  $x$  if  $a$  is greater (less) than zero.

THEOREM 4: If  $g(x) = \sum_{i=1}^k g_i(x)$  so that  $E[g(x)] = \sum_{i=1}^k E[g_i(x)]$ , then:

$$E[g(x)] C_g = \sum_{i=1}^k E[g_i(x)] C_{g_i} \quad (3.21)$$

where  $C_g$  and  $C_{g_i}$  are concentration indices for  $g(x)$  and  $g_i(x)$ , respectively.

Proof of the Theorem 4:

Substituting (3.13) in (3.12) gives:

$$C_g = 1 - \frac{2}{E[g(x)]} \int_0^{\infty} \sum_{i=1}^k F_1[g_i(x)] f(x) dx \quad (3.22)$$

which on interchanging the summation and integral signs becomes:

$$C_g = 1 - \frac{2}{E[g(x)]} \sum_{i=1}^k E[g_i(x)] \int_0^{\infty} F_1[g_i(x)] f(x) dx$$

Now  $C_{g_i}$  is defined as:

$$C_{g_i} = 1 - 2 \int_0^{\infty} F_1 [g_i(x)] f(x) dx \quad (3.24)$$

Substituting (3.24) in (3.23) and using the fact that  $E[g(x)] = \sum_{i=1}^k E[g_i(x)]$  gives the result (2.30). This proves the theorem.

Let us again assume that  $g(x) = a+bx$  so that  $E[g(x)] = a+b\mu$ . If  $b > 0$ ,  $g(x)$  is a monotonic increasing function, therefore the concentration index for  $g(x)$  will be same as the Gini-index of the function.

Now using the fact the Gini-Index of a constant is zero, and the Gini-Index of  $bx$  is same as the Gini-index of  $x$ , it follows from Theorem 4, that:

$$(a + b\mu) G^* = b\mu G \quad (3.25)$$

where  $G$  is the Gini-Index of  $x$  and  $G^*$  is the Gini-Index of the linear function  $x(a + bx)$ . We have the following corollary.

COROLLARY 8: If  $G$  is the Gini-Index of a random variable  $x$ , then the Gini-Index  $G^*$  of a linear function  $(a + bx)$  for  $b > 0$  is given by:

$$G^* = \frac{b\mu G}{a + b\mu} \quad (3.26)$$

where  $E(x) = \mu$ ,

In the above Corollary if  $a = 0$ ,  $G^* = G$  which implies that if all incomes are multiplied by a same constant, then the income inequality remains unchanged. Further,  $G^*$  is less (greater) than  $G$  if  $a$  is greater (less) than zero.

SOME APPLICATIONS OF THE THEOREMS

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In this section we shall consider some of the applications of the theorems given in the last section.<sup>4/</sup>

4.1 The Engel Curve

If  $g(x)$  is the equation of Engel curve of a commodity, then it follows from Corollary 1 and 2 that if its concentration curve lies above the egalitarian line, it is an inferior commodity, if the concentration curve lies between the Lorenz curve of  $x$  and the egalitarian line, it is a necessary commodity and if the concentration curve lies below the Lorenz curve, the commodity is luxury.

4.2 Consumption and Saving Functions

In the Keynesian case the consumption is related to income either linearly or curvilinearly. Let us first assume that the relation be linear:

$$c = \alpha + \beta x \quad (4.2.1)$$

where  $\beta$  is the marginal propensity to consume and  $x$  is the disposable income and  $c$  is the consumption expenditure of an individual. Since  $\alpha$  and  $\beta$  are greater than zero, it follows from Corollary 7 that the personal consumption expenditure is more equally distributed than the personal disposable income.

The saving function corresponding to (4.2.1) is:

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<sup>4/</sup> Many more applications of the theorems will be discussed in a forthcoming monograph which is under preparation.

$$s = -\alpha + (1 - \beta)x, \quad (4.2.2)$$

which again from Corollary 7 implies that the personal savings will be more unequally distributed than the personal disposable income provided the marginal propensity is less than one.

Let us now introduce the rate of interest as an additional variable in the savings function (4.2.2):

$$s = -\alpha + (1 - \beta)x + \lambda r, \quad (4.2.3)$$

where  $r$  is the rate of interest. If  $\beta < 1$ , then from Corollary 8 we obtain:

$$G_s = \frac{(1 - \beta)\mu G}{\mu_s} \quad (4.2.4)$$

where  $G$  and  $G_s$  are Gini-Indices of disposable income and savings, respectively.  $\mu$  is the mean disposable income and  $\mu_s$  is the mean saving which is given by:

$$\mu_s = -\alpha + (1 - \beta)\mu + \lambda r \quad (4.2.5)$$

Differentiating (4.2.4) with respect to  $r$  gives:

$$\frac{\partial G_s}{\partial r} = -\frac{(1 - \beta)\lambda \mu G}{\mu_s^2} \quad (4.2.6)$$

which leads to the conclusion that higher the interest rate, more equal will be the distribution of savings. This conclusion is of course based on the assumption that the increase in the interest rate does not alter the distribution of the disposable income.

Next we consider the curvilinear consumption and savings functions. If the average propensity to consume decreases as income rises, it is obvious that the income elasticity of consumption will be less than one and the income elasticity of saving will be greater than one. It immediately follows from Corollary 6 that the inequality of income is greater than that of spending but less than that of saving.

### 4.3 Effect of Direct Taxes on the Income Distribution

Let  $x$  be the pre-tax income of an individual and  $T(x)$  the tax function, then the disposable income is given by:

$$d(x) = x - T(x) \quad (4.3.2)$$

which is an increasing function of  $x$  if the marginal tax rate is less than one. Using Theorem 3 in (4.3.2) gives:

$$\mu_d F_1 [d(x)] = \mu F_1(x) - Q F_1 [T(x)] \quad (4.3.3)$$

where  $\mu$  and  $\mu_d$  are mean of pre-tax and post-tax income respectively and  $Q$  is the tax yield. The expression (4.3.3) can be written as:

$$F_1 [d(x)] - F_1(x) = \frac{Q}{\mu_d} [F_1(x) - F_1(T(x))] \quad (4.3.4)$$

If the tax function  $T(x)$  is progressive through out the income range, the tax elasticity will be greater than one for all  $x$  which from Corollary 2 implies that  $F_1(x) > F_1[T(x)]$  for all  $x$ . This leads to the conclusion that if the tax function is progressive, after-tax income distribution will be more equal than the before tax income distribution.



4.7 Taxation in an Inflationary Economy

Consider an economy in which prices and productivity are rising at annual rate of  $100 p$  and  $100 s$  percent. Suppose the incomes of all income units are increasing in the same proportion. Then income of a unit after  $t$

$$x(t) = [(1+p)(1+s)]^t x \quad (4.4.1)$$

where  $x$  is the initial income. Let the tax function be:

$$T(x) = a x^\delta, \quad (4.4.2)$$

then the tax collected at time  $t$  from an income unit with initial income  $x$  will be:

$$\begin{aligned} T(x(t)) &= a [(1+p)(1+s)]^{\delta t} x^\delta \\ &= [(1+p)(1+s)]^{\delta t} T(x) \end{aligned} \quad (4.4.3)$$

If

$$Q = \int_0^\infty T(x) f(x) dx \quad (4.4.4)$$

is the mean tax paid at time zero, then the mean tax paid at time  $t$  will be

$$Q(t) = [(1+p)(1+s)]^{\delta t} Q \quad (4.4.5)$$

which gives the average tax rate at time

$$\frac{Q(t)}{\mu(t)} = \frac{[(1+p)(1+s)]^{(\delta-1)t} Q}{\mu} \quad (4.4.6)$$

where  $E(x) = \mu$  and  $F_1[x(t)] = \mu(t)$ . Thus, if the taxes are progressive  $\delta > 1$ , the average tax rate will increase (decrease) over time if  $p, s$  are greater (less) than zero but less than one in absolute value.

The disposable income at time  $t$  of a unit having initial income  $x$  is  $x(t) - T\{x(t)\}$  and, therefore, applying Theorem 3 we obtain:

$$F_t^*(x) = \frac{1}{\mu^*(t)} \left[ \mu(t) F_1(x) - Q(t) q(x) \right] \quad (4.4.7)$$

where

$$q(x) = \frac{1}{Q} \int_0^x \alpha x^\delta f(x) dx \quad (4.4.8)$$

is the proportion of tax paid by income units having income less than or equal to  $x$  at time zero and  $F_t^*(x)$  is the proportion of the mean disposable income of the same income units at time  $t$ .  $\mu^*$  is the mean disposable income at time  $t$ :

$$\mu^*(t) = [(1+p)(1+s)]^t \mu - [(1+p)(1+s)]^{\delta t} Q \quad (4.4.9)$$

The equation (4.4.7) simplifies to:

$$F_t^*(x) - F_1(x) = \frac{1}{\mu^*(t)} \left[ (1+p)(1+s) \right]^{\delta t} Q [F_1(x) - q(x)] \quad (4.4.10)$$

If the tax function is progressive, i.e.  $\delta > 1$ , then from Corollary 2,  $F_1(x) > q(x)$  for all  $x$  which from (4.4.10) implies that the concentration curve for the disposable income at time  $t$  is higher than the Lorenz curve for income. Further, if the marginal tax rate is less than one, the disposable income is a monotonic increasing function of  $x$  which from Theorem 2 implies that the concentration curve for the disposable income at time  $t$  coincides with its Lorenz curve. Thus for a progressive tax system the after tax income at time  $t$  is more equally distributed than the before tax income.

Differentiating (4.4.10) with respect to  $p$  gives:

$$\frac{\partial F_t^*(x)}{\partial p} = \frac{(\delta - 1) t Q(t) \mu(t)}{(1 + p) \mu^2(t)} [F_1(x) - q(x)] \quad (4.4.11)$$

Again, if the tax system is progressive  $\delta > 1$  and  $F_1(x) > q(x)$  which implies the right-hand side of (4.4.11) is positive and, therefore, as  $p$  increases the Lorenz curve for after-tax income distribution will shift upward. Similarly, if the tax system is regressive,  $\delta < 1$  and  $F_1(x) < q(x)$  the right-hand side of (4.4.11) is again positive. The Lorenz curve shifts upward as  $p$  increases. Thus we can conclude that the inflation decreases the after tax income-inequality for both progressive and regressive tax systems provided the before tax distribution is not affected by inflation.

The above conclusion is valid only if the taxes are not adjusted to inflation. Suppose we change the tax rates every year by keeping  $\delta$  constant but change the parameter  $\delta$ . The tax function at time  $t$  can then be written as:

$$T_t(x) = \alpha_t x^\delta \quad (4.4.12)$$

where  $\alpha_t = \alpha$  at  $t = 0$ . Then the mean tax at time  $t$  will be:

$$Q(t) = \frac{\alpha_t}{\alpha} [(1+p)(1+s)]^{\delta t} Q \quad (4.4.13)$$

and, therefore, the average tax rate becomes:

$$\frac{Q(t)}{\mu(t)} = \frac{\alpha_t}{\alpha} \frac{Q}{\mu} [(1+p)(1+s)]^{(\delta-1)t} \quad (4.4.14)$$

Suppose we adjust  $\alpha_t$  every year such a way that the ratio of tax to income remains constant. Then from (4.4.14) it can be seen that

$$\alpha_t = \frac{\alpha}{[(1+p)(1+s)]^{(\delta-1)t}} \quad (4.4.15)$$

which means  $\alpha_t$  is to be reduced every year if the tax function is progressive and for a regressive tax function  $\alpha_t$  should be increased.

Now using (4.4.15) in (4.4.10) gives:

$$F_t^*(x) - F_1(x) = \frac{Q}{\mu - Q} [F_1(x) - q(x)] \quad (4.4.16)$$

which implies that  $dF_t^*(x) / dx = 0$ . Thus we conclude that if the tax function is adjusted every year such a way that the tax-income ratio is constant every year, then the inflation will not change the after tax income distribution for any tax system progressive or regressive.

Let:

- $G_t^*$  = Gini-index of the after-tax distribution at time  $t$ .
- $G$  = Gini-index of before-tax income at  $t=0$ .
- $C$  = Concentration index of taxes paid at time zero.

Then from Theorem 4 we obtain:

$$G_t^* = G - \frac{Q(t)}{\mu(t) - Q(t)} (C - G) \quad (4.4.17)$$

which gives the elasticity of the Gini-index of the after tax distribution with respect to inflation rate as:

$$\frac{p}{G_t^*} \frac{\partial G_t^*}{\partial p} = \frac{-pt(\delta - 1) \mu(t) Q(t) (C - G)}{(1 + p) [\mu(t) - Q(t)]^2 G_t^*} \quad (4.4.18)$$

We can now compute the Gini-index and the elasticity of the Gini-index with respect to inflation rate. The source of data used for this purpose is the Australian Taxation Statistics for the assessment year 1971-72 (Income tax year 1970-71). The data are available in grouped form. The income considered is the actual income for individual tax payers less the expenditure incurred in gaining that income.

Gini-index of before tax income was computed to be .3456 and for the tax paid the concentration index was .5419. The tax function was estimated to be:<sup>5/</sup>

<sup>5/</sup> The weighted regression method was used to estimate the tax function.

$$\log T = -6.8068 + 1.588 \log x \quad (4.4.19)$$

where  $x$  represents income and  $T$  taxes. The squared correlation between estimated and actual values of  $T$  was computed to be .98.

Table I presents the Gini-index of the after-tax income and its elasticity with respect to the rate of inflation. It is to be noted the Gini-index is quite sensitive to the inflation and the sensitivity increases with the rate of inflation and also over time.

Table 1: GINI-INDEX OF THE AFTER TAX INCOME AND ITS ELASTICITY WITH RESPECT TO INFLATION RATE

Rate of Inflation	1970 - 1971		1971 - 1972		1972 - 1973		1973 - 1974	
	Gini-Index	Elasticity	Gini-Index	Elasticity	Gini-Index	Elasticity	Gini-Index	Elasticity
-10	.3124	0.00	.3141	.0076	.3157	.0143	.3171	.0201
- 5	.3124	0.00	.3129	.0038	.3134	.0074	.3138	.0109
- 3	.3124	0.00	.3124	.0022	.3124	.0045	.3125	.0067
0	.3124	0.00	.3117	0.0000	.3110	0.0000	.3103	0.0000
2	.3124	0.00	.3110	-.0022	.3096	-.0047	.3081	-.0071
5	.3124	0.00	.3106	-.0037	.3086	-.0080	.3065	-.0128
10	.3124	0.00	.3094	-.0074	.3061	-.0165	.3024	-.0279
15	.3124	0.00	.3083	-.0110	.3035	-.0250	.2981	-.0455

#### 4.5 INCOME INEQUALITY BY FACTOR COMPONENTS

Suppose the total family income  $x$  is written as the sum of  $n$  factor incomes  $x_1, x_2, \dots, x_n$ , then from Theorem 4, we obtain

$$G = \frac{1}{\mu} \sum_{i=1}^n \mu_i c_i \quad (4.5.1)$$

$C_i$  is the concentration index of the  $i$ -th factor income component which has mean income  $\mu_i$ . This equation expresses the Gini-index of the total family income as the weighted average of the concentration indices of each factor income component, the weights being proportional to the mean income of each component.

The equation (4.5.1) can be used to analyze the contribution of inequality of each factor income to the total inequality.<sup>6/</sup> To illustrate this numerically we utilize the data obtained from the Australian Survey of Consumer Expenditure and Finance, 1967-68.<sup>7/</sup> The results are presented in Table 2. It is seen from the table that the income from employment, i.e., wages and salaries contribute 82.68% to the total inequality. Unincorporated business income is second contributing 11.38% and the property income, i.e. interest, dividend and rent contribute only 3.24% to the total inequality.

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\* This problem has also been considered by , Ranis and o [ 3 ] and Pyatt [ 4 ].  
<sup>7/</sup> See Podder and Kakwani [ 8 ].



Table 2: INEQUALITY BY FACTOR INCOME COMPONENTS

Factor Income	Mean Income	Concentration Index	Contribution of each factor of income inequality to the total income inequality	% of cash contribution
Employment	3399 (96.5)	.3449	.3014	82.68
Unincorp.	276 (7.7)	.5852	.0415	11.38
Property	106 (2.9)	.4322	.0118	3.24
Regular Annuity	37 (1.0)	.0544	.00052	.14
Capital items	18 (.4)	.3737	.00169	.46
Capital gains	39 (1.1)	.7737	.007776	2.13
Miscellaneous	14 (.3)	.2922	.001037	.28
Total income	3890	.3645	.3645	100.00

#### 4.6 THE LINEAR EXPENDITURE SYSTEM

The demand equations of the linear expenditure system (LES) are given by

$$v_i = p_i \gamma_i + \beta_i (v - a) \quad (4.6.1)$$

where  $v_i = p_i q_i$  is the per capita expenditure for the  $i$ -th commodity,  $p_i$  is its price and  $q_i$  is the per capita quantity demanded.  $v = \sum_{i=1}^n p_i q_i$  is total per capita expenditure and  $a = \sum_{i=1}^n p_i \gamma_i$  is the subsistence expenditure.

$\beta_i$  is interpreted as the marginal budget share of the  $i$ -th commodity.

The above system of demand equations is derived by maximizing the Klein and Rubln [4] form of the utility function.

$$u = \sum_{i=1}^n \beta_i \log(q_i - \gamma_i) \quad (4.6.2)$$

in which the  $\beta$ 's and  $\gamma$ 's are parameters with  $0 < \beta_i < 1$ ,  $\sum_{i=1}^n \beta_i = 1$ ,  $\gamma_i \geq 0$  and  $q_i - \gamma_i > 0$ .

Let  $G_i$  be the Gini-index for the distribution of the expenditure on the  $i$ -th commodity and  $G^*$  be the Gini-index for the total expenditure, then using Corollary 6 on the equation (4.6.1) we obtain

$$G_i = \frac{\beta_i \mu^* G^*}{\mu_i} \quad (4.6.3)$$

where  $\mu^*$  is the mean total expenditure and  $\mu_i$  is the mean expenditure on the  $i$ -th commodity. This equation can also be written as

$$G_i = \eta_i G^* \quad (4.6.4)$$

where  $\eta_i$  is the expenditure elasticity at the mean expenditures. Thus the expenditure elasticity of the  $i$ -th commodity at the mean expenditures is equal to the ratio of the Gini-indices of the distributions of the  $i$ -th commodity

expenditures and the total expenditure respectively. If the elasticity is greater (less) than one, the expenditure on the  $i$ -th commodity is more (less) unequally distributed than the total expenditure.

#### 4.5.1 Income Inequality and Pr —

We now consider the effect of price changes on the income inequality of the real income.

Substituting (4.6.1) into (4.6.2), we obtain the indirect utility function as

$$u = \sum_{i=1}^n \beta_i \log \beta_i + \log(v-a) - \sum_{i=1}^n \beta_i \log p_i \quad (4.6.5)$$

Suppose the prices  $p_i$  change to  $p_i^*$ , and the total expenditure  $v$  changes to  $v^*$ , then the resulting change in the utility will be

$$\Delta u = \log(v^*-a^*) - \log(v-a) - \sum_{i=1}^n \beta_i (\log p_i^* - \log p_i) \quad (4.6.6)$$

where  $a^* = \sum_{i=1}^n p_i^* \gamma_i$ . If the change in utility is set to zero, we obtain the total per capita expenditure  $v^*$  in order that the family maintains the same utility:

$$v^* = a^* + (v-a) \prod_{i=1}^n \left( \frac{p_i^*}{p_i} \right)^{\beta_i} \quad (4.6.7)$$

$v^*$  will be the real expenditure. Let  $G_R$  be the Gini-index of the real expenditure, then apply Corollary 8 on this equation gives

$$G_R = \frac{\prod_{i=1}^n \left( \frac{p_i^*}{p_i} \right)^{\beta_i} \mu^* G^{***}}{a^* + (\mu^* - a) \prod_{i=1}^n \left( \frac{p_i^*}{p_i} \right)^{\beta_i}} \quad (4.6.8)$$

where  $G^*$  is the Gini-index of the money expenditure in the base year.

It is obvious from the equation (4.6.8) that if all the prices change in the same proportion  $G_R = G^*$  i.e., the inequality of the distribution of the money expenditure in the base year is same as the inequality of the real expenditure.

The ratio  $\frac{v^*}{v}$  is the true cost of living index.<sup>8/</sup> It converts the money expenditure into real expenditure. In the spirit of true cost of living index, we propose to use the ratio  $\frac{G_R}{G^*}$  as an index of the income inequality to take into account the effects of relative price changes. This index converts the inequality of the money household expenditure distribution to the inequality of the real household expenditure. If this index is less than one, it implies that the relative price changes are making the expenditure distribution more unequal.

The numerical results on the index of income inequality are presented in Table 3. The U-K data was used for this purpose.<sup>9/</sup> It is seen from the table that the relative price change from 1964 to 1972 have the effect of increasing income inequality. The 1971-72 change is particularly marked.

#### 4.6.2 Income Inequality and Prices: An Alternative Approach

Suppose the price of  $j$ -th commodity changes by  $\alpha_j$  percent, then the demand for the  $i$ -th commodity will change by  $\eta_{ij}\alpha_j$  percent, where  $\eta_{ij}$  is the price elasticity of the  $i$ -th commodity with respect to  $j$ -th price. The resulting demand for the  $i$ -th commodity becomes

$$q_i^* = q_i(1 + \eta_{ij}\alpha_j) \quad (4.6.3)$$

<sup>8/</sup> See Klein and Rubin [4].

<sup>9/</sup> See Muellbauer [5] for the detailed description of the data.

**Table 3:** INDEX OF INCOME INEQUALITY IN U.K. 1964-72

Year	Over 9 Goods		True cost of living Index	Index of Income Inequality
	$a^*/a$	$\Pi \left( \frac{P_i^*}{P_i} \right)^{y_i}$		
1964	1.000	1.000	1.000	1.000
1965	1.048	1.041	1.046	.9952
1966	1.090	1.074	1.085	.9898
1967	1.118	1.100	1.112	.9892
1968	1.168	1.156	1.164	.9931
1969	1.229	1.210	1.223	.9894
1970	1.296	1.278	1.290	.9907
1971	1.399	1.375	1.391	.9885
1972	1.481	1.443	1.469	.9823

Table 4: PERCENTAGE CHANGE IN GINI-INDEX WITH INCREASE IN PRICE OF EACH GOOD BY 10%

Number	Good	% Change In Gini-Index
1	Food	1.821
2	Clothing	.033
3	Housing	-.148
4	Durables	-.180
5	Others	-1.524

Table 6 gives the percentage change in the Gini-index of the real expenditure when the price of each commodity has increased by 10% at a time. It is seen that the price increase of food and clothing increases the inequality of real expenditure while the increase in price of three other goods decrease the inequality of the real expenditures.

where

$$\begin{aligned} \eta_{ij} &= -\frac{\beta_i}{v_i} p_i \gamma_i \quad \text{if } i \neq j \\ &= \frac{\beta_j}{v_j} (p_j \gamma_j + v-a) \quad \text{if } i = j \end{aligned} \quad (4.6.10)$$

The expenditure on the  $i$ -th commodity at base year prices will be

$$\begin{aligned} v_i^* &= v_i \left(1 - \frac{\beta_i}{v_i} p_j \gamma_j \alpha_j\right) \quad \text{if } i \neq j \\ &= v_j \left[1 - \frac{\alpha_j \beta_j}{v_j} (p_j \gamma_j + v-a)\right] \quad \text{if } i = j \end{aligned} \quad (4.6.11)$$

The total expenditure is then obtained as:

$$\tilde{v}^* = (1 - \alpha_j \beta_j) v + \alpha_j (a \beta_j - p_j \gamma_j) \quad (4.6.12)$$

where the use has been made of the restriction  $\sum_{i=1}^n \beta_i = 1$ .

Let  $\tilde{G}_R$  be the Gini-index of the real expenditure, then applying Corollary on the equation (4.6.12) gives

$$\tilde{G}_R - G^* = \frac{-\alpha_j (\beta_j a - p_j \gamma_j) G^*}{\mu^* - \alpha_j \gamma_j p_j - \alpha_j \beta_j (\mu^* - a)} \quad (4.6.13)$$

The expression (4.6.13) provides the percentage change in the Gini-index of the real expenditure when the price of the  $j$ -th commodity changes by  $\alpha_j \%$ , other prices remaining constant.

For the numerical illustration we used the data obtained from the Mexico Household Survey conducted by the Bank of Mexico in 1968. The families considered were urban entrepreneurs. The parameters of the linear expenditure system were estimated using individual observations.

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