A Model for Analyzing Lenders' Perceived Risk

Optimal International Borrowing, Capital Allocation, and Credit Worthiness Control

A model for analysing lenders' perceived risk

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Bank lending involves, in most cases, the risk that the borrower will not be able or willing to honour his obligations. The existence of default risk is an important factor in explaining the observed behaviour of lenders as demonstrated in the works of Jaffee and Modigliani (1969), Smith (1972), Azzi and Cox (1976), Jaffee and Russell (1976), and others. Lenders' behaviour in this case depends crucially on their subjective evaluation of the probability of default. Thus, to explain lending behaviour, knowledge of lenders' subjective information is generally unobservable; and empirical analysis of lending behaviour is, therefore, difficult. Furthermore, there is often reason to believe that subjective information may vary considerably from lender to lender or from transaction to transaction because of previous experience, personal relationships, etc.; and, hence, the role of subjective perceptions cannot be ignored.

The purpose of this paper is to develop a model which facilitates inference about lenders' subjective default probabilities. Several alternative approaches of constructing both point and interval estimates of subjective default probabilities are proposed under various assumptions about competition, loss rate distribution, risk aversion, and relative loan size.

The assumption which makes these approaches possible is that lending takes place with a fixed probability of default. That is, lending transactions are assumed to be of sufficiently negligible size relative to the borrower's scale of operations (e.g. lending to sovereign borrowers) so that the probability of default is not influenced by the lender's current decision, i.e. the interest rate on the loan does not affect default probability. This assumption differs from that employed in the theoretical papers cited above and, hence, removes complications associated with endogenizing the probability of default. Nevertheless, the simplification which leads to the empirical possibilities developed in this paper is plausible in many situations—particularly in international lending. For example, suppose the state of São Paulo, Brazil, borrows $100 million in the Eurodollar market with the guarantee of the...
federal government of Brazil. The likelihood of default in this case would surely be affected only slightly by the lender's decision since the overall debt of Brazil is $32 billion dollars (Newsweek, 1978).

I. THE MODEL

Consider a simple one-period model similar to the one presented by Smith (1972). The lender is assumed to have a given amount, $W$, of loanable funds, all or part of which can be lent to the particular borrower under consideration at interest rate $\bar{r}$. The interest rate is exogenously given if the lender operates within a competitive capital market, or it depends on the volume of loans granted (according to the borrower's demand schedule) if the lender has some degree of monopoly power. Alternatively, part or all of the lender's funds can be lent at interest $\rho$ which is the opportunity cost of capital and which is considered risk free.\(^2\)

The borrower's demand schedule is given by the relation,

\[ P = f(L), \quad P' < 0 \text{ in the monopolist's case} \]
\[ P' = 0 \quad \text{for the competitive lender}, \]

where $L$ denotes the volume of loans granted. It will be useful to define the risk premium, $r$, which is the difference between the lender's opportunity cost of capital and the interest rate actually paid by the borrower, namely,

\[ \bar{r} = r + \rho. \]  

Using Equation 1 and considering $\rho$ as a constant, the demand schedule can be defined for the risk premium $r$,

\[ r = r(L), \quad r' = \bar{r}' \leq 0, \]

where, as before, $\bar{r}' = 0$ applies to the case of the competitive lender.

The borrower is assumed to possess (at the time of loan negotiations) assets totalling $K$ which were financed in part by previous borrowing and in part from the borrower's own resources. His total outstanding debt as of the time of the loan under consideration is given by $D$. Following Smith (1972), it is assumed that the liquidity situation of the borrower at the time of loan maturity is related to borrower resources (including the new loan) by a stochastic, linear homogenous function of the form $\gamma x [K + L]$, where $\gamma$ is a continuous random variable with probability density function $f(\gamma)$ defined over the interval $(0, \infty)$. The parameters of $f$ are influenced by various borrower characteristics as viewed by the lender; and $f(\gamma)$ reflects, therefore, the latter's subjective assessment of borrower prospects.

When the loan under consideration becomes due, the borrower must pay $(1 + \bar{r}) \times L$ in addition to other obligations which will be due on his older outstanding debt $D$, say $\phi \times D$ where $\phi$ denotes the rate of interest and amortization payments; $\phi$ is not affected by the new lender.

\(^2\)As in the earlier works on lending under uncertainty cited above, portfolio considerations are ignored for the sake of simplicity.
A model for analysing lenders' perceived risk

Obviously, if $\gamma$ will be sufficiently high so as to maintain the inequality

$$\gamma \times (K + L) - (1 + \dot{r}) \times L - \phi \times D \geq 0,$$

then all creditors will be paid in full and no default takes place. Using Equation 4, one can define the critical $\gamma$ value, (say, $\gamma^*$)

$$\gamma^* = \frac{(1 + \dot{r}) \times L + \phi \times D}{K + L}.$$

If $\gamma < \gamma^*$, not all obligations can be met, and a partial default takes place. In such a case, it is assumed that creditors are paid according to the proportion of debt they hold. In particular, the lender under consideration will be paid

$$\gamma \times (K + L) \times \frac{L}{(L + D)}.$$

The present value of the new lender's wealth (say, $y$) is thus given by

$$y = \begin{cases} 
W - L + \frac{(1 + \dot{r})}{1 + \rho} \times L & \text{if } \gamma \geq \gamma^* \\
W - L + \gamma \times \frac{K + L}{(1 + \rho)} \times \frac{L}{(L + D)} & \text{if } \gamma < \gamma^*. 
\end{cases}$$

Rearranging the terms in Equation 6, present value $y$ can be rewritten as

$$y = \begin{cases} 
w + r \times \theta \times L & \text{if } \gamma \geq \gamma^* \\
w - h \times L & \text{if } \gamma < \gamma^*. 
\end{cases}$$

where

$$\theta \equiv \frac{1}{1 + \rho},$$

$$h \equiv 1 - \gamma \times \frac{K + L}{(L + D) \times (1 + \rho)},$$

and

$$r \equiv \text{risk premium as defined in Equation 2.}$$

Note that $h$ is random since it is a transformation of the random variable $\gamma$. From the formulation in Equation 7, it follows that, for a given value of $\gamma$, $h$ is the rate of loss incurred by the lender. The range of values over which $h$ is defined is $(0, \gamma^*)$. Thus, using the definition of $\gamma^*$ as given in Equation 5, $h$ is confined between upper and lower bounds $l$ and $h$, respectively, where

$$h \equiv \left\{ \left[ \frac{1 - \phi}{(1 + \rho)} \right] \times D = \left[ \frac{r}{(1 + \rho)} \right] \times L \right\} / (L + D).$$

It is implicitly assumed that no priority debt exists. However, one could assume that priority debt is paid first and that the other creditors share the remaining liquidity according to their share in regular debt. The appropriate analysis, however, remains unchanged.
Denote the lender's utility function by \( U \), where \( U \) is defined over present value of wealth. The utility function is assumed to have the properties

\[
U' > 0, \quad U'' \leq 0,
\]

where \( U'' < 0 \) implies risk aversion and \( U'' = 0 \) reflects risk neutrality. It is further assumed that, in the case \( U'' < 0 \), relative risk aversion is no greater than one, i.e.,

\[
-(U''/U') \times y \leq 1.
\]

This is a plausible assumption as argued by Arrow (1971, p. 98). The lender's objective is to maximize the expected utility of present-value wealth by an optimal choice of loan size, i.e. using Equation 6b

\[
\text{Max}_L \Pi = \int_{\gamma^*}^{\infty} U(W + r \times \theta \times L) \times f(\gamma) \, d\gamma + \int_0^{\gamma^*} U(W - h \times L) f(\gamma) \, d\gamma.
\]

Note, however, that

\[
\int_{\gamma^*}^{\infty} U(W + r \times \theta \times L) f(\gamma) \, d\gamma = U(W + r \times \theta \times L) \times [1 - P(\gamma^*)],
\]

where

\[
P(\gamma^*) = \int_0^{\gamma^*} f(\gamma) \, d\gamma
\]

represents the probability of default. Note, also, using the definitions of \( h \) and \( h \), that

\[
\int_0^{\gamma^*} U(W - h \times L) f(\gamma) \, d\gamma \equiv \int_h^1 U(W - h \times L) \times p(h) \, dh
\]

where

\[
p(h) = f \left[ (1 - h)(1 + \rho) \times \frac{(L + D)}{(K + L)} \right].
\]

The objective function in Equation 8 can thus be written as

\[
\text{Max}_L \Pi = [1 - P(\gamma^*)] \times U(W + r \times \theta \times L) + \int_h^1 U(W - hL) \times p(h) \, dh.
\]

In the general case, the probability of default \( P(\gamma^*) \), as well as the specific loss rate density \( p(h) \), depends on the size of loan granted by the current lender and on the rate of interest charged for the loan (as inspections of \( \gamma^* \) and \( h \) verify). However, as will be shown below, in the case where the relevant loan size is small, relative to the size of the borrower's assets (i.e. when \( L/K \) is small), the probability of default and the distribution of loss rates can be treated as given (from the point of view of the current lender). This simplifies the analysis considerably and facilitates several applications as will be demonstrated in the following sections.

In order to verify the assertions regarding \( P(\gamma^*) \) and \( p(h) \), define the loan to asset ratio \( \hat{l} \) (i.e. \( \hat{l} = L/K \)). Rewriting the definitions of \( \gamma^* \), \( h \), and \( h \) by simply dividing numerators and
denominators by $K$, one obtains

\[ \gamma^* \equiv \frac{(1 + \hat{r}) \times \hat{l} + \phi \times (D/K)}{1 + \hat{l}} \]

\[ h \equiv 1 - \gamma \times \frac{(1 + \hat{l})}{[l + (D/K)](1 + \rho)} \]

and

\[ h \equiv \left\{ \left[ \frac{1 - \phi}{(1 + \rho)} \right] \times \left( \frac{D}{K} \right) - \frac{\hat{l} \times (1 + \rho)}{[l + (D/K)]} \right\}. \]

Taking the limit as $\hat{l} \to 0$ yields

\[ \gamma^* \bigg|_{\hat{l} = 0} = \phi \times \frac{D}{K}; \]

\[ h \bigg|_{\hat{l} = 0} = \frac{1 - \gamma}{(1 + \rho) \times (D/K)}; \]

\[ h \bigg|_{\hat{l} = 0} = \frac{1 - \phi}{1 + \rho}. \]

Thus, when $\hat{l}$ is sufficiently small, $\gamma^*$ and $h$ approach fixed parameters from the current lender's point of view, and the (random) loss rate $h$ is independent of the size of the current loan ($L$) and of the risk premium ($r$) charged on the loan.

These results imply that, with the assumption of negligible $\hat{l}$, the probability of default $P(\gamma^*)$ and the density of the loss rate $p(h)$ are not appreciably affected by the actions of the current lender and can be treated as fixed parameters in the lender's problem. Rather, the dominant factors underlying these probabilities are the existing economic prospects facing the borrower.

Before proceeding to characterize the optimal solution, one further modification needs to be made—namely, extending the model to a loan duration of $N$ years. Assuming that, at most, one default takes place within a loan's duration and that loans are repaid in 'balloon' form (interest payments paid annually and the principal repaid in the last year), the formulation of the objective function (Equation 9) can be retained provided that $\theta$ is redefined as

\[ \theta \equiv \frac{1 - (1 + \rho)^{-N}}{\rho} \]

(note that one can show that

\[ \sum_{t=1}^{N} \hat{r} \times L \times (1 + \rho)^{-t} + L \times (1 + \rho)^{-N} - L = r \times \theta \times L, \]

and $h$ is reinterpreted as a discounted rate of loss. This is possible since both $\rho$ and $N$ are viewed as exogenous by the lender with the former parameter determined by market opportunities and the latter being dictated to a large extent by the lender's and borrower's overall liquidity projections (Beim, 1977, p. 723).

For notational convenience in the following analysis, define

\[ P = P(\gamma^*), y_1 \equiv W + r \times \theta \times L, \text{ and } y_h \equiv W - h \times L. \]
Differentiation (Equation 9) with respect to \( L \), the following first-order condition characterizes the optimal solution:

\[
\frac{d \Pi}{d L} = (1 - P) \times \theta \times U'(y_1) \times (r + Lr') - \int_{h}^{1} \ell''(y_h) \times h \times p(h) \, dh = 0. \tag{10}
\]

The second-order condition requires

\[
\frac{d^2 \Pi}{d L^2} = (1 - P) \times U''(y_1) \theta^2 (r + Lr')^2 + \int_{h}^{1} U''(y_h) h^2 p(h) \, dh \\
+ (1 - P) \times \theta \times U'(y_1) \times (2r' + Lr'') < 0. \tag{11}
\]

From Equation 11, it is obvious that risk aversion is a necessary and sufficient condition for

\[
\frac{d^2 \Pi}{d L^2} < 0
\]

in the case of a competitive lender \( (r' = r'' = 0) \). In the case of a risk neutral monopolistic lender,

\[
(2r' + Lr'') < 0
\]

is necessary and sufficient to guarantee concavity of the objective function. With risk aversion and monopoly, the latter condition is not necessary but remains sufficient. Henceforth, it will be assumed that second-order conditions hold for the lender under consideration.

II. COMPARATIVE STATIC RESULTS

Before proceeding to demonstrate the wide range of empirical uses of this model, it is first useful to examine the various comparative static properties of the model for the purpose of showing that the model is indeed plausible.

Demand elasticity

Considering the monopolistic lender, Equation 10 implies that the lender operates on the nonelastic portion of the borrower's demand schedule. To see this, note that \( r + Lr' = r \times (1 - \eta) \), where \( \eta \) is the absolute value of demand elasticity at the optimal point. Obviously, if \( \eta > 1 \), Equation 10 cannot hold since the left-hand side is negative.

Default probability

A borrower who is more risky (i.e., who carries a higher probability of default) will be granted less credit whether the lender is competitive or not. This can be confirmed by assuming that, for any given rate of loss \( h \), the relative likelihood is at least as great as before, \( d \, p(h) \geq 0 \) for \( h \leq h \leq 1 \).

Hence, using the fact that

\[
P = \int_{h}^{1} p(h) \, dh,
\]
A model for analysing lenders' perceived risk

one finds that

\[ \int_{h}^{1} d\rho(h) > 0 \]

if \( d\rho > 0 \). By differentiating Equation 10, one then obtains

\[ \frac{dL}{d\rho} = \left[ \frac{\partial^2 \Pi}{\partial L^2} \right]^{-1} \times \int_{h}^{1} \left[ U'(y_1) \times (r + Lr') + U'(y_h) \times h \right] \frac{d\rho(h)}{d\rho} < 0, \tag{12} \]

where the sign is established using Equation 11 and the earlier result that \( r + Lr' > 0 \) for both a monopolist and a competitive lender. This result in Equation 12 implies, in the case of a monopolist, that riskier borrowers are charged a higher risk premium. This is also the case in a competitive market since, with a higher probability of default, a smaller amount of loans will be offered by any individual lender in the market at any given interest rate. The aggregate supply of loans to the borrower under consideration will thus decline. With a negatively sloped demand for loans and a positively sloped aggregate supply, a decline in supply must yield a higher equilibrium level of risk premium (with a lower amount of loans in a competitive market is indeed positively sloped (i.e., that supply is increasing with higher interest rates). This is done by differentiating Equation 10:

\[ \frac{dL}{dr} = - \left( \frac{\partial^2 \Pi}{\partial L^2} \right)^{-1} (1 - P) \times \theta \times U'(y_1) \times \left[ 1 + \frac{U''(y_1)}{U'(y_1)} \right] r0L. \tag{13} \]

Under the assumption that relative risk aversion is no greater than one, the term in square brackets on the right-hand side of Equation 13 is positive; and it is thus concluded that

\[ dL/dr > 0. \]

It should be noted in the case where the lender considers the loan to have an impact on the probability of default, however, that the offer curve by each individual lender is backward bending irrespective of attitudes toward risk as shown in Jaffee and Modigliani (1969), Smith (1972), and Azzi and Cox (1976). The different results in the present analysis are due to the fact that a higher interest rate increases the marginal expected utility while, in the earlier models, a higher interest rate may reduce expected utility by causing a higher probability of default.

Risk aversion

Considering the role of risk aversion in the model, it is intuitively expected that, when lenders are more risk averse, the volume of loans will be lower and risk premiums will be higher. This can be shown simply by assuming a specific form of the utility function such that risk aversion is reflected in a single parameter. The two most common such utility functions are the constant relative risk-aversion family of functions and the constant absolute risk-aversion family of functions. These are given, respectively, by

\[ U(y) = ay^{1-\alpha}, \quad 0 < \alpha < 1 \tag{14a} \]

\[ U(y) = a - e^{-\alpha y}, \quad \alpha > 0 \tag{14b} \]
where $\alpha$ is a parameter of risk aversion such that the higher $\alpha$ is associated with higher risk aversion. Assuming the case of Equation 14a, differentiation of Equation 10 obtains

$$\frac{dL}{d\alpha} = \left[ \frac{\partial^2 \Pi}{\partial L^2} \right]^{-1} \times (W + r\theta L)^{-\alpha} \times \left[ (1 - P) \times \theta \times (r + Lr') \right]$$

$$\times \int_{h}^{1} h \times (W - hL)^{-\alpha} \times p(h) \ln \left( \frac{W - hL}{W + r\theta L} \right) \times dh \right]. \tag{15a}$$

Since

$$\ln \left( \frac{(W - hL)}{(W + r\theta L)} \right) < 0,$$

it follows that

$$\frac{dL}{d\alpha} < 0.$$

Similarly, assuming the case of Equation 14b and differentiating Equation 10 yields

$$\frac{dL}{d\alpha} = \left[ \frac{\partial^2 \Pi}{\partial L^2} \right]^{-1} \times \left[ \int_{h}^{1} h \times p(h) \times e^{-\alpha(W - hL)} \times (h + r\theta) \times L \times dh \right] < 0. \tag{15b}$$

The results in Equations 15a and b imply that a higher degree of risk aversion causes a smaller volume of loans and higher risk premiums in both competitive and noncompetitive markets.

### III. SUGGESTED APPLICATIONS

The results in the previous section are plausible and intuitive and imply that the equilibrium relation in Equation 10 can be usefully considered for further expositional and empirical purposes. In this section several such empirical applications are demonstrated. For this purpose, suppose that lenders' information and evaluation distinguish between the overall probability that a default will take place and the (conditional) probability that a given rate of loss ($h$) will be incurred. Hence, for conceptual purposes, and so that subjective default probability can be meaningfully discussed, suppose that the probability density function of loss rate $h$ is proportional to the overall default probability such that the conditional probability of $h$ depends on $h$ alone and not on the economic factors which explain the overall probability. This implies

$$\frac{p(h)}{P} = \psi(h), \quad \int_{h}^{1} \psi(h) \, dh = 1, \tag{16}$$

where $\psi$ depends only on $h$ irrespective of the economic factors which determine $P$. Note, that, with this assumption, the (unconditional) probability of any given loss rate increases when the overall default probability increases, as one would expect, since Equation 16 implies

$$p(h) = \psi(h) \times P.$$
A model for analysing lenders' perceived risk

thus,
\[
\frac{dp(h)}{dp} = \psi(h) > 0.
\]

Applying Equation 16 in Equation 10 and rearranging yields
\[
r = \frac{1}{(1 - \eta)} \left( \frac{P}{(1 - P)} \frac{h}{\theta} R \right)
\]
where
\[
\hbar \equiv \int_{h}^{1} \psi(h)h \, dh
\]
and
\[
R = \frac{\int_{h}^{1} h \times \psi(h) \times U'(y_{h}) \, dh}{\hbar \times U'(y_{1})}.
\]

For a monopolistic lender, Equation 17 demonstrates that the risk premium includes a demand elasticity factor, \((1 - \eta)^{-1}\); the odds of default, \(P/(1 - P)\); a time-effect factor, \(\theta^{-1}\); an average loss-rate factor \(\bar{h}\); and a risk-aversion factor, \(R\). The latter factor is identically equal to one under risk neutrality in which case
\[
U'(y_{h}) = U'(y_{1}).
\]

With risk aversion, it is easy to show that
\[R > 1\] and \(\partial R/\partial L > 0\).

Equation 17 offers a great deal of flexibility in empirical inference of subjective default probabilities because of its simplicity. This flexibility is demonstrated in the following three applications.

Threshold default probability

A simple application of the model is related to the concept of 'threshold probability' for competitive lenders. The threshold probability is the highest value of default probability (say, \(P^*\)) for which a loan will be granted to any borrower and depends on \(r\), \(\theta\), and \(\bar{h}\). Borrowers with probability lower than \(P^*\) will be given credit with the amount of credit increasing in the ratio
\[
\frac{[P^*/(1 - P^*)]}{[P/(1 - P)]}.
\]

Using Equation 17, the threshold probability is calculated by noting that, at \(P = P^*\), it must hold that \(L = 0\) which implies that
\[
U'(y_{1}) = U'(y_{h}) = U'(W).
\]
This obtains

\[ P^* = \frac{r}{(\bar{h}/\theta) + r}. \]  

(18)

By differentiating Equation 18, one can verify that the threshold probability is positively related to the risk premium and the loan duration (i.e.

\[ \frac{\partial P^*}{\partial r} > 0, \frac{\partial P^*}{\partial N} > 0 \]

while being negatively related to the average loss rate and to the opportunity cost of capital (i.e.

\[ \frac{\partial P^*}{\partial \bar{h}} < 0, \frac{\partial P^*}{\partial \rho} < 0. \]

Using Equation 18 in 17, with \( \eta = 0 \), further yields

\[ R(L) = \frac{P^*/(1 - P^*)}{P/(1 - P)}. \]  

(19)

The right-hand side of Equation 19 is the ratio of odds evaluated at the threshold and actual values of probability. From Equation 19, one can derive the result that loans (and the risk aversion premium) increase as the right-hand side of Equation 19 increases.

A useful property of the threshold probability is that it is independent of the degree of risk-aversion as is apparent from Equation 18. Since the values of \( \rho \), \( N \), and \( \bar{h} \) are either known or can be estimated without major difficulty, one can calculate the threshold probability which applies to different risk premiums. This approach is demonstrated in Table 1 for \( \rho = 0.065 \) and \( \bar{h} = 0.15, 0.2 \). The results are presented in terms of \( q^* \) (the short-run threshold probability) so as to allow comparison between cases with different loan durations. That is, if at most, one default can take place within the duration of the loan, then the probability \( P \) of default over the entire period of the loan is

\[ P = 1 - (1 - q)^N, \]

provided that \( q \), the probability of default in any given year, is constant.\(^4\) It is reasonable to expect that, given current data, lenders assume \( q \) to remain essentially constant since no data beyond the period in which the loan is granted are available for projection. The \( q^* \) in Table 1 is a threshold or upper bound on \( q \) and is related to \( P^* \) just as \( q \) is related to \( P \).

Using this approach, one can very easily confine the subjective probability \( q \) to the interval \([0, q^*]\) under competition by simply using data on \( r, \rho, N \) and \( \bar{h} \) (note that \( \theta \) is determined by \( \rho \) and \( N \)); moreover, the interval \([0, P^*]\) is fairly narrow even for small \( \bar{h} \). Furthermore, if \( \bar{h} \) is unknown but can be bounded from below by \( \bar{h}^* \), then \( P \) can be confined to the interval \([0, P^{**}]\) where

\[ P^{**} = \frac{r}{(\bar{h}^*/\theta) + r} \]

(since \( P^* \) is decreasing in \( \bar{h} \)) or \( q \) can be confined to an associated interval \([0, q^{**}]\) where \( P^{**} = 1 - (1 - q^{**})^N \). Hence, the simple transaction data on \( r, \rho, \) and \( N \) is sufficient

\(^4\)The reader will note that this definition of \( P \) was indeed in the description of the model.
A model for analysing lenders' perceived risk

Table 1. Threshold short-run probabilities and risk premiums for 5- and 10-year loans*

<table>
<thead>
<tr>
<th>Loan duration (years)</th>
<th>r (per cent)</th>
<th>q* (h = .15)</th>
<th>q* (h = .2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>N = 5</td>
<td>.25</td>
<td>.013</td>
<td>.010</td>
</tr>
<tr>
<td>N = 10</td>
<td>.25</td>
<td>.011</td>
<td>.009</td>
</tr>
<tr>
<td>N = 5</td>
<td>.50</td>
<td>.026</td>
<td>.020</td>
</tr>
<tr>
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<td>.50</td>
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<tr>
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<td>N = 10</td>
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<td>.024</td>
</tr>
<tr>
<td>N = 5</td>
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<td>.042</td>
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</tr>
<tr>
<td>N = 10</td>
<td>2.00</td>
<td>.065</td>
<td>.053</td>
</tr>
</tbody>
</table>

*Assuming the opportunity cost of capital p is 6.5 per cent and N = 5, the value of θ is 4.156, with N = 10, the value of θ is 7.1888.

...to develop some information about the magnitudes of subjective default probabilities which might exist.

An application in inferring lenders' subjective probabilities: The case of the Eurodollar market

Another application of the model in this paper can be made by inferring interval estimates of lenders' subjective probabilities of default from a minimal amount of observed market data. To demonstrate this possibility, 20 observations on loans granted in the Eurodollar market during the third quarter of 1973 have been used. These observations (presented in the Appendix) refer to loans made during that period to public (or publicly guaranteed) entities in developing countries for which data on risk premium and loan duration were available.5

5The short period covered was chosen so as to avoid any significant variations in the opportunity cost of capital among observations. The estimates are meant solely to be illustrative. There is no presumption that these estimates are the actual (objective) probabilities nor that the World Bank in any way concurs with these perceptions.
The procedure employed here uses the assumptions that the market operates competitively and that the subjective loss-rate distribution is singular in order to establish bounds on the lenders’ perceived default probabilities. The first assumption implies that $\eta = 0$, and the second implies that $h$ is a constant, say, $\tilde{h}$. Assuming that the utility function can be specified by the constant elasticity formulation (Equation 14b), the risk-aversion premium $R$ can then be written

$$ R = \left( \frac{1 + r\theta l}{1 - \tilde{h}} \right) ^\alpha, $$

where $l = L/M$ is the share of the loan in total loanable funds and $\alpha$ is the relative risk-aversion parameter ($0 < \alpha < 1$).

While the 20 observations on Eurodollar loans range in loan duration between 10 and 15 years, the probability concept which is of interest is again one that is free of the loan duration effect. Thus, combining Equations 17 and 20 and the relation between $q$ and $P$ the following formula is obtained:

$$ q = 1 - (1 - P)^{1/N} = 1 - \left[ 1 + \frac{r\theta}{\tilde{h}} \left( \frac{1 - \tilde{h}l}{1 + r\theta l} \right) ^\alpha \right]^{-1/N}. $$

To make use of Equation 21 in the Eurocurrency market, it can be observed that the average rate of loss in international banking is low; in almost all cases of debt-service problems, loans are renegotiated and rescheduled. Hence, for exemplary purposes, a value of $\tilde{h} = .1$ is used. As for $l$, $\rho$, and $\alpha$, it can be noted from Equation 21 that

$$ \partial q/\partial l, \partial q/\partial \rho, \text{and} \partial q/\partial \alpha < 0. $$

Thus, specifying sets of upper and lower bounds on these coefficients leads to lower and upper bounds, respectively, for the short-run probability $q$. For example, consider upper bounds of $l = .3$, $\rho = .08$, and $\alpha = .8$ and lower bounds of $l = 0$, $\rho = .06$, and $\alpha = 0$. These values seem to span the range of reality. That is, $l = 0$ and $\alpha = 0$ are the lowest possible values (assuming no risk-loving behaviour), while $l = .3$ and $\alpha = .8$ seem to be above and beyond all likely possibilities. No bank seems to loan anything near 30 per cent of its funds to one borrower; also $\alpha$ close to 1, in the case of the constant elasticity utility function, implies an extremely high degree of risk aversion. Finally, the opportunity cost of capital seems to be between 6 and 8 per cent. Using these bounds, the associated interval estimates for subjective default probability (as perceived by Eurodollar lenders) in Table 2 are possible for the Eurodollar market transactions considered above. However, these estimates do not necessarily imply judgments on the objective (true) probability.

Several interesting observations can be made on the basis of these results. First, the upper and lower bounds span a range for the probability of each case within .8 of 1 per cent or less. This is a small interval relative to the overall variation in Table 2. Second, the calculated $q$s for those countries with more than one observation (e.g., Algeria) are fairly close for most

4In this respect, Friedman (1977, p. 55) noted: ‘Losses in U.S. banks’ overseas operations have been less, both in absolute terms and as a proportion of total risk assets, than in U.S. operations.’ A similar observation has been made by Beim (1977, p. 717).

5Normally, one would want to estimate $\tilde{h}$ since the estimates are somewhat sensitive to this particular parameter.
A model for analysing lenders' perceived risk

Table 2. Perceived short-run probabilities in the Euromarket third quarter, 1973

<table>
<thead>
<tr>
<th>Observation number</th>
<th>Country</th>
<th>q upper bound</th>
<th>q lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Algeria (N=15)</td>
<td>.044</td>
<td>.039</td>
</tr>
<tr>
<td>2</td>
<td>Algeria (N=12)</td>
<td>.047</td>
<td>.042</td>
</tr>
<tr>
<td>3</td>
<td>Algeria (N=10)</td>
<td>.048</td>
<td>.044</td>
</tr>
<tr>
<td>4</td>
<td>Brazil (N=10)</td>
<td>.052</td>
<td>.049</td>
</tr>
<tr>
<td>5</td>
<td>Brazil (N=12)</td>
<td>.049</td>
<td>.044</td>
</tr>
<tr>
<td>6</td>
<td>Colombia</td>
<td>.043</td>
<td>.039</td>
</tr>
<tr>
<td>7</td>
<td>Gabon</td>
<td>.079</td>
<td>.072</td>
</tr>
<tr>
<td>8</td>
<td>Greece</td>
<td>.040</td>
<td>.036</td>
</tr>
<tr>
<td>9</td>
<td>Iran (N=12)</td>
<td>.037</td>
<td>.033</td>
</tr>
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<td>10</td>
<td>Iran (N=10)</td>
<td>.037</td>
<td>.033</td>
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<td>Korea (N=10)</td>
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<tr>
<td>12</td>
<td>Korea (N=11)</td>
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</tr>
<tr>
<td>13</td>
<td>Mexico (N=12)</td>
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</tr>
<tr>
<td>14</td>
<td>Mexico (N=10)</td>
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<td>.031</td>
</tr>
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<td>15</td>
<td>Nicaragua</td>
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<td>16</td>
<td>Peru</td>
<td>.075</td>
<td>.068</td>
</tr>
<tr>
<td>17</td>
<td>Philippines</td>
<td>.087</td>
<td>.079</td>
</tr>
<tr>
<td>18</td>
<td>Senegal</td>
<td>.083</td>
<td>.075</td>
</tr>
<tr>
<td>19</td>
<td>Zaire</td>
<td>.079</td>
<td>.072</td>
</tr>
<tr>
<td>20</td>
<td>Zambia</td>
<td>.069</td>
<td>.063</td>
</tr>
</tbody>
</table>

a Calculated with α = 0, l = 0, ρ = .06, and h = .1.
b Calculated with α = .8, l = .3, ρ = .08, and h = .1.

of these cases (theoretically, they should be identical). Also, Peru, Zaire, and Zambia, which seem to have been considered as higher risks by Euromarket lenders as indicated by their q value, have indeed experienced serious debt-service problems in years after 1973. These observations may serve as an indication of the creditability of the assumptions. In summary, it appears that fairly precise estimates of default probabilities, as perceived by lenders, are possible under competition using only the terms of the loan if loss rate for the event of default can be determined (estimated).

Possible econometric applications

In addition to the above examples, Equation 17 also has interesting econometric possibilities. For the purpose of econometric work, it is useful to consider the common logistic specification for the probability, namely,

\[
P = \frac{e^{\beta_0 + \beta'X}}{1 + e^{\beta_0 + \beta'X}}
\]

(22)
where \( \beta_0 \) is a constant, \( \beta \) is a vector of coefficients, and \( X \) is a vector of economic indicators including loan duration which are considered by lenders as relevant risk indicators. With this specification, the odds are log linear, i.e.,

\[
\ln \left[ \frac{P}{(1 - P)} \right] = \beta_0 + \beta'X.
\]

Hence, Equation 17 becomes

\[
\ln r = -\ln (1 - \eta) + \ln \bar{h} - \ln \theta + \beta_0 + \beta'X + \epsilon.
\]

where

\[
\epsilon = \ln R = \ln \left[ \int_{y_0}^{h \psi(h) U'(y_\eta) \, dh} [\bar{h} U'(y_1)] \right] - \ln [\bar{h} U'(y_1)].
\]

The model in Equation 23 has several interesting potential simplifications. First, for risk-neutral lenders, it is clearly the case from above that \( \epsilon = \ln R = 0 \); hence,

\[
\ln r = -\ln (1 - \eta) + \ln \bar{h} - \ln \theta + \beta_0 + \beta'X.
\]

With the presence of a disturbance term, Equation 23 can be estimated in a variety of situations to determine which factors are considered important by lenders in assessing default probabilities. For instance, \( \eta, \bar{h}, \) and \( \theta \) can be calculated \( \theta \) depends only on \( p \) and \( N \) which are often recorded, and \( \bar{h} \) is also occasionally recorded for different classes of borrowers) and then included in a regression estimating \( \beta_0 \) and \( \beta \) where a coefficient of 1 or \(-1\) is imposed for \( \ln (1 - \eta) \).\( \ln \bar{h}, \) and \( \ln \theta \) as needed according to Equation 24. Alternatively, one may find data where combinations of \( \eta, \bar{h}, \) and \( \theta \) are held constant and can thus be included in a constant term. Another approach (Feder and Just, 1977a) is to consider some of the first right-hand terms as randomly distributed among borrowers and then use a variance-components approach in estimating \( \beta \) from a time series of cross-section data.

Finally, an interesting observation can be made for Equation 23 in the general case of risk preferences. As it stands, the \( \epsilon \) term in Equation 23 would be very difficult to treat econometrically; besides, it would imply inclusion of the dependent variable \( r \) on the right-hand side. But in a broad range of cases, it turns out that \( \epsilon \) is negligible relative to both \( \ln r \) and the usual variations in \( \ln r \) among observations. This can be demonstrated assuming a constant elasticity utility function in which case Equation 20 applies. Using the same plausible limits on parameters used to generate Table 2, together with additional limits on \( r \) of \( 0 \leq r \leq 0.025 \) and on \( N \) of \( 0 \leq N \leq 10 \), it can be shown through simple calculations that \( 1 \leq R < 1.07 \) and hence \( 0 < \ln R < 0.068 \). By comparison, as \( r \) varies from 0.00575 to 0.025 (which seems to span the observed range of premiums in the Eurodollar market), the dependent variable \( \ln r \) varies from \(-3.698\) to \( 5.159 \). Hence, even with most conceivable limits on \( l, \rho, \) and \( \alpha \), it is thus clear that \( \epsilon \) is negligible and can, for most practical purposes, be included with any disturbance in Equation 23.

Considering a competitive market, Equation 24 can become fairly simple once \( \epsilon \) is incorporated in the error term if competition implies \( \eta = 0 \). The equation is then reduced to

\[
\ln r = \ln \bar{h} - \ln \theta + \beta_0 + \beta'X + \nu
\]
where $v$ is a random disturbance. This formulation is exemplified in the following estimation example which pertains to transactions in the Eurodollar market in the period January, 1975, using a set of 29 observations on public or publicly guaranteed loan transactions (corresponding to 23 developing countries) as reported in Appendix 2. Reliable indications in the period covered imply that the cost of capital ($\rho$) was quite stable. For instance, the Eurodollar three-month and six-month deposit rates (which are closely related to the cost of capital) ranged from 6 per cent to 7 per cent and 7 per cent to 8 per cent, respectively, while in the previous year they fluctuated substantially from 8.5 per cent to 14 per cent. Adopting the assumption that $\rho$ was approximately constant throughout the period covered thus simplifies the treatment of $\theta$ in Equation 25 since one can experiment with a number of likely values of $\rho$ (say, .06, .07, and .08) in alternative estimates. The vector $X$ includes, in addition to loan duration (which is expected to be positively related to default probability), three variables which have been suggested as potentially related to debt-servicing capacity of developing countries and, hence, to the risk of default (Frank and Cline, 1971, and Feder and Just, 1977b). These variables are:

1. The ratio of debt-service payments to exports (also referred to as the debt-service ratio). Higher ratios reflect a heavier burden of debt service payments and are thus expected to be positively related to the probability of default.

2. The ratio of imports to G.N.P., which reflects claims of foreign exchange which are competing with debt-service obligations. Higher import to G.N.P. ratios should thus be positively related to the probability of default.

3. The level of G.N.P. per capita as a proxy for degree of development and the ability to divert resources away from consumption in order to service external debt. Higher per capita G.N.P. is thus negatively related to default risk.

Data for the calculation of these variables were obtained from various World Bank and International Monetary Fund publications. Having defined the vector $X$ and using three alternative values for $\rho$ for the calculations of $\theta$, the corresponding three regression estimates of Equation 25 are reported in Table 3.

The results are practically identical for the different values of $\rho$ and thus imply that lack of information regarding the exact cost of capital is not serious. All parameters are statistically significant at a 7.5 per cent one-sided significance level, and their signs confirm the a priori arguments advanced. Thus, longer loan maturities, higher debt-service ratios, and higher import to G.N.P. ratios increase the perceived probability of debt-service problems of borrowing governments while higher per capita incomes reduce the perceived probability.

To examine one useful implication of results of this type, note that the constant term reported in Table 3 is an estimate of $\ln \bar{h} + \beta_0$ (plus the mean of the error term $v$ if the latter has a nonzero mean). For given values of $\bar{h}$ (and mean error when applicable), one can calculate the corresponding $\beta_0$ values. These, together with the $\beta$ coefficients reported in Table 3, can be used in calculating perceived probabilities using Equation 22, where each alternative value of $\beta_0$ corresponds to a set of probabilities (Feder and Just, 1977a).

Using the parameters in Table 3, one can also establish a direct link between terms of
credit and economic performance criteria which can be useful for a borrower in managing economic resources wisely. Consider, for example, a five-year loan sought by an economy with a G.N.P. of $1000 per capita. Table 4 presents the risk premiums corresponding to various debt-service ratio and import to G.N.P. ratio combinations. Using the results of this analysis, a borrowing country can thus evaluate the interaction of, say, import policy and borrowing policy with respect to its terms of credit. In a similar way, one can represent the risk premiums corresponding to other combinations of explanatory variables.

Table 3. Regression results with Eurodollar data: January–July, 1975

<table>
<thead>
<tr>
<th>Variable</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan duration (years)</td>
<td>.153</td>
<td>.149</td>
<td>.145</td>
</tr>
<tr>
<td></td>
<td>(6.515)*</td>
<td>(6.341)</td>
<td>(6.171)</td>
</tr>
<tr>
<td>Debt-service ratio</td>
<td>.467</td>
<td>.467</td>
<td>.467</td>
</tr>
<tr>
<td></td>
<td>(1.671)</td>
<td>(1.672)</td>
<td>(1.673)</td>
</tr>
<tr>
<td>Import to G.N.P. ratio</td>
<td>.432</td>
<td>.431</td>
<td>.431</td>
</tr>
<tr>
<td></td>
<td>(2.511)</td>
<td>(2.510)</td>
<td>(2.510)</td>
</tr>
<tr>
<td>Per capita G.N.P. ($1000)</td>
<td>-.089</td>
<td>-.088</td>
<td>-.088</td>
</tr>
<tr>
<td></td>
<td>(1.541)</td>
<td>(1.541)</td>
<td>(1.541)</td>
</tr>
<tr>
<td>Constant</td>
<td>-3.496</td>
<td>-3.502</td>
<td>-3.509</td>
</tr>
</tbody>
</table>

\( R^2 \)

<table>
<thead>
<tr>
<th></th>
<th>.69</th>
<th>.68</th>
<th>.67</th>
</tr>
</thead>
</table>

\( \bar{R}^2 \)

<table>
<thead>
<tr>
<th></th>
<th>.64</th>
<th>.63</th>
<th>.61</th>
</tr>
</thead>
</table>

*Figures in parentheses are 't' ratios.

Table 4. Risk premiums for a five-year loan by a borrowing country with $1000 per capita G.N.P.

<table>
<thead>
<tr>
<th>Debt-service ratio</th>
<th>Import to G.N.P. ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>.1</td>
</tr>
<tr>
<td>.1</td>
<td>.0155</td>
</tr>
<tr>
<td>.2</td>
<td>.0163</td>
</tr>
<tr>
<td>.3</td>
<td>.0170</td>
</tr>
</tbody>
</table>

*The calculations are based on Case 2 of Table 3, but almost identical results are obtained with Cases 1 and 3.
IV. CONCLUSION

This paper presents a model of lending under default risk. With an underlying assumption that the probability of default is determined by factors not controlled by the lender (which is essentially the case when a new loan is small relative to the borrower's existing debt), the effects of risk, risk aversion, and expected loss rate on the equilibrium levels of loan supply and risk premiums are analysed for both competitive and noncompetitive markets. The model is shown to be useful for assessing subjective default probabilities from market data, using the Eurodollar market as an example. The model also enables the determination of upper bound threshold probabilities for acceptable customers, given the relevant duration of loan and interest rate. Further utilization of the model for various econometric purposes is discussed in detail, and it is shown that in many cases the complicated term reflecting risk aversion can be ignored since its magnitude and range of variation are negligible. This simplifies greatly the procedures for econometric application of the model as demonstrated in an example utilizing additional Eurodollar market data.

Each of these approaches can provide useful information depending on data availability. If one simply observes the risk premium, cost of capital, loan duration, and average loss rate in the event of default, then the lender's subjective probability can be confined to an interval \((0, P^*)\). If, in addition, bounds can be established on the share of the individual loan of the lender's total loanable funds and on the lender's risk aversion, the subjective probabilities can be confined to a much narrower interval. Finally, if information can alternatively be gained on the economic factors which influence a lender's subjective probability, then regression techniques can be used to produce point estimates of lender's subjective probabilities under certain conditions. Results of the latter type show particular promise for a borrower in managing credit worthiness since they provide information on how the borrower's other economic decisions affect its cost of capital.

APPENDIX

Table 1A. Transactions in the Eurodollar market: third quarter, 1973*  

<table>
<thead>
<tr>
<th>Borrowing country</th>
<th>Risk premium (per cent)</th>
<th>Loan duration (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algeria</td>
<td>1.008</td>
<td>15</td>
</tr>
<tr>
<td>Algeria</td>
<td>.938</td>
<td>12</td>
</tr>
<tr>
<td>Algeria</td>
<td>.875</td>
<td>10</td>
</tr>
<tr>
<td>Brazil</td>
<td>1.00</td>
<td>12</td>
</tr>
<tr>
<td>Brazil</td>
<td>.955</td>
<td>10</td>
</tr>
<tr>
<td>Colombia</td>
<td>.75</td>
<td>10</td>
</tr>
<tr>
<td>Gabon</td>
<td>1.75</td>
<td>10</td>
</tr>
<tr>
<td>Greece</td>
<td>.75</td>
<td>12</td>
</tr>
<tr>
<td>Iran</td>
<td>.688</td>
<td>12</td>
</tr>
</tbody>
</table>
Table 1A. (cont.)

<table>
<thead>
<tr>
<th>Borrowing country</th>
<th>Risk premium (per cent)</th>
<th>Loan duration (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iran</td>
<td>.625</td>
<td>10</td>
</tr>
<tr>
<td>Korea</td>
<td>1.205</td>
<td>11</td>
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<tr>
<td>Korea</td>
<td>1.5</td>
<td>10</td>
</tr>
<tr>
<td>Mexico</td>
<td>.659</td>
<td>12</td>
</tr>
<tr>
<td>Mexico</td>
<td>.575</td>
<td>10</td>
</tr>
<tr>
<td>Nicaragua</td>
<td>1.325</td>
<td>10</td>
</tr>
<tr>
<td>Peru</td>
<td>1.6</td>
<td>10</td>
</tr>
<tr>
<td>Philippines</td>
<td>2.0</td>
<td>10</td>
</tr>
<tr>
<td>Senegal</td>
<td>1.875</td>
<td>10</td>
</tr>
<tr>
<td>Zaire</td>
<td>1.75</td>
<td>10</td>
</tr>
<tr>
<td>Zambia</td>
<td>1.425</td>
<td>10</td>
</tr>
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</table>


*When several premiums were charged in different stages of the loan duration period, a weighted average was calculated. Also, when several loans with the same duration were observed, a weighted average was calculated.*

Table 2A. Transactions in the Eurodollar market: January–July, 1975*

<table>
<thead>
<tr>
<th>Borrowing country</th>
<th>Risk premium (per cent)</th>
<th>Loan duration (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algeria</td>
<td>1.375</td>
<td>7</td>
</tr>
<tr>
<td>Argentina</td>
<td>2.0</td>
<td>5.5</td>
</tr>
<tr>
<td>Bolivia</td>
<td>2.179</td>
<td>8</td>
</tr>
<tr>
<td>Bolivia</td>
<td>2.0</td>
<td>5</td>
</tr>
<tr>
<td>Brazil</td>
<td>1.75</td>
<td>5</td>
</tr>
<tr>
<td>Brazil</td>
<td>1.875</td>
<td>7</td>
</tr>
<tr>
<td>China</td>
<td>1.5</td>
<td>6</td>
</tr>
<tr>
<td>Colombia</td>
<td>1.75</td>
<td>7</td>
</tr>
<tr>
<td>Costa Rica</td>
<td>2.0</td>
<td>4.75</td>
</tr>
<tr>
<td>Ecuador</td>
<td>2.0</td>
<td>5</td>
</tr>
<tr>
<td>Ecuador</td>
<td>1.5</td>
<td>6</td>
</tr>
<tr>
<td>El Salvador</td>
<td>1.75</td>
<td>5</td>
</tr>
</tbody>
</table>
### Risk Loan Borrowing Premium Duration (per cent) (years)

<table>
<thead>
<tr>
<th>Borrowing country</th>
<th>Risk premium (per cent)</th>
<th>Loan duration (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Guyana</td>
<td>2.5</td>
<td>5</td>
</tr>
<tr>
<td>Indonesia</td>
<td>1.725</td>
<td>5</td>
</tr>
<tr>
<td>Ivory Coast</td>
<td>1.875</td>
<td>5</td>
</tr>
<tr>
<td>Jamaica</td>
<td>1.9</td>
<td>5</td>
</tr>
<tr>
<td>Korea</td>
<td>2.0</td>
<td>5</td>
</tr>
<tr>
<td>Malaysia</td>
<td>1.625</td>
<td>5</td>
</tr>
<tr>
<td>Mexico</td>
<td>1.5</td>
<td>3</td>
</tr>
<tr>
<td>Mexico</td>
<td>1.5</td>
<td>5</td>
</tr>
<tr>
<td>Nicaragua</td>
<td>2.07</td>
<td>5</td>
</tr>
<tr>
<td>Panama</td>
<td>1.75</td>
<td>5</td>
</tr>
<tr>
<td>Panama</td>
<td>1.875</td>
<td>7</td>
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<tr>
<td>Peru</td>
<td>1.788</td>
<td>7</td>
</tr>
<tr>
<td>Philippines</td>
<td>1.755</td>
<td>6</td>
</tr>
<tr>
<td>Spain</td>
<td>1.5</td>
<td>5</td>
</tr>
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<td>Spain</td>
<td>1.5</td>
<td>7</td>
</tr>
<tr>
<td>Sudan</td>
<td>1.75</td>
<td>5</td>
</tr>
<tr>
<td>Zambia</td>
<td>2.0</td>
<td>5</td>
</tr>
</tbody>
</table>


*When several premiums were charged in different stages of the loan duration period, a weighted average was calculated. Also, when several loans with the same duration were observed, a weighted average was calculated.*

### REFERENCES


"Official borrowers like any others, wish to borrow on the best available terms in the private market. . . After all, they do command the reserves of the country as well as access to domestic capital markets and the central bank. Such factors enter in their balance of payments management and, more particularly, their external debt and reserve management. They know that their external debt management strategies must be played out in the market place." (Friedman 1977 a, p. 16)

The determination of appropriate international borrowing policy has attracted the attention of many economists because economic growth, particularly in developing countries, depends heavily on the inflow of foreign funds. Work on this topic dates back to Adam Smith; but more recent contributions have been made by Hamada (1966, 1969), Bardhan (1967), Van Long (1974), Manning (1972), and McCabe and Sibley (1976). A common approach in related economic analyses has involved a "small country" assumption that has been taken to imply an infinitely elastic supply of loans at a given interest rate. It has been argued recently by Hanson (1974), however, that the supply of loans may not be completely elastic even for the small country because, as debts get large, the probability of a debt crisis for the individual country increases. That is, as debts get large relative to the country (not the market), it becomes more likely that the borrower will not be able to make repayments on schedule; hence, lenders will become more reluctant to lend as the expected, discounted returns decline. As indicated by Mohammed and Saccomanni (1973), there is no doubt that this risk plays a major role in the lending practices of commercial banks. Non-profit-oriented lending institutions are also likely to be concerned with the probability that reschedulings will be required; hence, their lend-
ing decisions may also be influenced to some degree by their borrowers' credit status (Cline and Sargan (1975)). It thus follows that even small countries do not face an infinitely elastic supply of foreign funds; rather, the terms of credit are endogenous depending on credit-worthiness. Credit-worthiness, in turn, is determined by the countries' economic performance (usually measured by a few accepted economic indicators).

Since economic-performance indicators are affected by economic policies, it is important to explore borrowing policies for optimal growth, taking into account the relationship between economic performance and terms of credit. Although empirical results suggest that several economic variables are related to debt servicing capacity, one may accept the use of only a few indicators as a useful simplification for theoretical analysis (Feder and Just (1977 a, b)). Hanson has suggested the use of the debt-equity ratio for this purpose. However, in the one-sector model developed by Hanson, this approach implicitly leads to the assumption that "resources can always be costlessly allocated to obtain the necessary foreign exchange for debt servicing" [11, p. 619, footnote 10]. Hence, the only factor limiting the availability of foreign exchange is the overall productive capacity of the country which may be represented by the capital stock.

Once installed, however, capital is not a malleable factor than can be equally productive in any sector. Although some types of capital are more flexible than others (transportation equipment, power plants, etc.), it is more likely with many exports, particularly in developing countries where exports are composed mainly of nonmanufactured goods, that investments are highly specialized (mines, dams, etc.). Even when reallocation of capital is possible, the time lag involved may be too long for lenders to consider the productive capacity of other sources as relevant for the evaluation of export potential in the case of debt payment difficulty. Since the export sector is the main source of foreign exchange earnings in many countries and capital reallocation may not be possible, it thus seems that the most appropriate case for developing countries may be where the terms of credit depend specifically on the size of the export sector (as well as the debt burden). This is consistent with the findings of a recent study regarding lending behavior in the Eurodollar market where two variables related to ex-

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1 Such policies have been suggested by observers of international markets; see Irvine et al. (1970), Friedman (1977 b).
port performance (the ratio of debt service to export earnings and a measure of export variability) appear to be considered as relevant by lenders, while the ratio of debt to gross national product (which is a proxy for the debt-equity ratio) was found insignificant (Feder and Just 1977 a).

Another balance of payments item which is of importance in creditworthiness evaluations is the size of foreign exchange reserves, which is usually viewed in comparison to the import bill (Feder and Just (1977 a), Goodman (1977)). That rational reserve policies are carried out with reference to credit worthiness (among other arguments) is evident from Friedman (1977 a, p. 18) who states: “Other developing countries, like Brazil, are borrowing both to finance external deficits and to rebuild reserves, profiting from the experience that a strong, visible level of reserves improves creditworthiness as well as providing an additional cushion to meet contingencies.”

The purpose of the present paper is to consider a two-sector model where export and import activities are defined explicitly. The model recognizes lenders' concern regarding default on international debt through a credit-worthiness dependent supply function of foreign funds. It is assumed that capital, once allocated between the export or nonexport sector, cannot be reallocated². Accordingly the creditworthiness assessment carried out by lenders (and which focuses on the foreign exchange attributes of the economy) is reflected in an interest rate which is positively related to the volume of anticipated debt service payment relative to the magnitude of export earnings. Similarly, the interest rate is negatively related to the size of foreign exchange reserves relative to the volume of import expenditures.

The model presents a characterization of optimal investment and borrowing decisions and discussed their policy implications in a mixed economy where both governmental and private decision making coexist.

The Model

The model below describes as small, two-sector open economy. One sector produces an export good, while the other sector produces a good which can be used for either consumption or investment.

² This assumption has been adopted in several other optimal growth models where terms of credit and optimal borrowing are not the major interest; see works by Bose (1968), Chakravarty (1969), Dasgupta (1969) Johansen (1967), and Ryder (1969).
can be augmented with imports of capital goods. The economy has access to international capital markets, but the terms of credit it obtains depend on its credit-worthiness status. The planning authority is assumed to be interested in maximizing the discounted utility streams of consumption.

The following notation is used in the analysis. All variables should carry a time subscript, which is omitted for the purpose of brevity. A dot over a variable denotes a rate of change over time.

A. Variables

(i) Flow variables

\[ F = \text{A composite good which can be used for both consumption and/or investment;} \]
\[ G = \text{Export good (not used domestically);} \]
\[ C = \text{Consumption;} \]
\[ I_d = \text{Capital formation from domestic output;} \]
\[ I = \text{Total investment;} \]
\[ M = \text{Quantity of imports of capital goods;} \]
\[ L = \text{Gross loans;} \]
\[ U = \text{Utility of consumption.} \]

(ii) Stock variables

\[ K_F, K_G = \text{Stock of capital in } F \text{ and } G \text{ sectors, respectively;} \]
\[ B = \text{Outstanding external debt;} \]
\[ R = \text{International reserves.} \]

(iii) Other variables

\[ x = \text{The proportion of total investment which is directed to the } F \text{ sector;} \]
\[ \tau = \text{Interest rate on external debt.} \]

B. Parameters

\[ i = \text{Rate of interest earned on foreign exchange reserves;} \]
\[ q = \text{The planner's rate of time discount;} \]
\[ \theta = \text{Rate of amortization of external debt;} \]
\[ \delta = \text{Rate of capital depreciation;} \]
\[ P_G = \text{International price of the export good;} \]
\[ P_m = \text{International price of the capital good;} \]
\[ \bar{r} = \text{Average interest rate as perceived by lenders.} \]

C. Model equations

(1) \[ G = G (K_F) ; \ G' > 0 ; \ G'' < 0 . \]
(2) \[ F = F (K_G) ; \ F' > 0 ; \ F'' < 0 . \]
The above equations describe the production technologies of the two sectors, which utilize capital only. Labor is assumed a non-binding constraint. For simplicity, labor growth is ignored.

\[ F(K_t) = C + I_d. \]

Equation (3) describes the allocation of the \( F \) good output between consumption and investment.

\[ \tau = \tau \left[ P_G - (\nu + \bar{r})B, R - P_m M \right]; r_1 \equiv -\frac{\partial \tau}{\partial \left[P_G - (\nu + \bar{r})B\right]} < 0; \]
\[ r_2 \equiv -\frac{\partial \tau}{\partial \left(R - P_m M\right)} < 0; \quad \tau (\infty, \infty) = i. \]

Equation (4) defines the endogenous interest rate on external debt as a (declining) function of two indicators of debt servicing capacity: (i) The difference between export foreign exchange earnings and the average volume of debt service payments (the latter being approximated by some perceived average rate of interest \( \bar{r} \), which lenders expect to prevail on the average); and, (ii) The difference between international reserves held by the economy and the import bill.

These two indicators correspond to two concepts widely used in credit worthiness assessment by lenders in international capital markets: (i) the debt service pressure measure which compares the major source of foreign exchange (exports) to the most rigid expenditure item in the balance of payment (debt service); and, (ii) the import-coverage measure which compares the size of reserves to the volume of imports, thereby reflecting the extent to which imports can be financed by reserve depletion if, for some unexpected reason, foreign currency inflows (exports and new loans) are not available.

While in most cases these measures are expressed as ratios (rather than differences), it is more convenient to follow the present formulation. The economic essence of the analysis is unaffected. As both common sense and empirical evidence suggest, the interest rate will be

\[ \text{Evidence on the prevalence of debt service and import coverage measures in credit worthiness analysis is available in many studies. See for instance Goodman (1977) who reports the results of a survey conducted among American banks engaging in international lending. Similar evidence is provided in Feder and Just (1977 a) who analysed factors affecting credit terms in the Eurodollar market. See also the articles by Friedman (1977 a), Brackenridge (1977), Puz (1977) and van-Agtmael (1976), all of whom are senior executives in international banks.} \]
lower for borrowers with higher exports (relative to debt service) and higher reserves (relative to imports), thus the partial derivatives $r_1$ and $r_2$ are negative. Also, for a risk-free borrower (such that both arguments of the $r$ function tend to infinity), the rate of interest will approach the going rate on international deposits ($i$).

\[ U = U(C) \ ; \ U' > 0 \ ; \ U'' < 0 \ ; \ U'(0) = \infty . \]

The utility of consumption has the standard properties of utility functions, namely, positive but decreasing marginal utility, and infinite marginal utility at $C = 0$.

\[ I = I_d + M . \]

Total investment is composed of domestic capital formation augmented by imports of capital goods. While the present model assumes that imports are composed of capital goods only, one can show that identical results may be derived from a model where imports are composed of consumption goods.

\[ \hat{R} = (P_G G + L + iR) - [P_m M + (r + \delta) B] . \]

The difference between foreign exchange inflows (composed of exports, loans and interest earnings on reserves) and foreign exchange outflows (composed of imports and debt service payments) equals the change in international reserve holdings.

\[ \dot{K}_f = \alpha I - \delta K_f . \]

\[ \dot{K}_p = (1 - \alpha) I - \delta K_p . \]

Equations (8) and (9) present the rate of change (over time) of capital stocks in the $F$ and $G$ sectors (i.e., net investments) as the difference between the new investments allocated to each sector and the volume of capital depreciation$^4$.

\[ \dot{B} = L - \delta B . \]

The change in indebtedness is equal to the difference between gross loans and amortization payments.

$^4$ The proportion of investment allocated to each sector should be constrained between zero and one. It will be assumed that this condition holds at the relevant range.
The planner's objective is maximization of discounted utility of consumption. Assuming an infinite horizon the problem can be formulated as

\[
\text{Max}_{\alpha, M, L, C} \int_{0}^{\infty} U(C) \, dt.
\]

subject to initial stock values and equations (1) - (10).

Problem (11) can be solved using the Pontryagin Maximum Principle. Denote the present value Hamiltonian by \(H\) and define the constant variables \(\lambda, \mu, \eta, \text{ and } \varphi\), which are dynamic shadow prices of the stock variables \(K_f, K_g, B\) and \(R\), respectively. Then, using the relation \(I = F(K_f) - C + M\) (from (3) and (5)), one can write

\[
H \cdot e^{-\alpha t} = U(C) + \lambda \{F(K_f) - C + M - \delta K_f\} + \mu \{(1 - \alpha) [F(K_f) - C + M] - \delta K_g\} + \eta (L - \delta B) + \varphi \{[P_g G(K_g) + L + iR] - [P_m M + (r + \delta) B]\}
\]

It should be noted that the non-negativity requirement for \(C\) is not specified since the assumption \(U'(0) = \infty\) eliminates such a solution. The non-negativity of \(M\) and \(L\), and the requirement \(0 < \alpha < 1\) are assumed to be non-effective constraints at the range of the optimal path described in the present paper. It thus follows that the optimality conditions to be derived below are applicable only at a certain range of the optimal path (including the steady state). Other portions of the optimal path include corner solutions (e.g., zero investment in one sector or the other, zero borrowing, etc.) and seem to us to be less interesting from a practical point of view. Similarly, existence and uniqueness of the solution depend on additional constraints which are not spelled out.

The optimality conditions require (in addition to appropriate transversality conditions), at the range where all controls are positive:

\[
\frac{\partial H}{\partial C} = 0 \quad \Rightarrow U' - \alpha \lambda - (1 - \alpha ) \mu = 0
\]

\[
\frac{\partial H}{\partial M} = 0 \quad \Rightarrow \alpha \lambda + (1 - \alpha) \mu - \varphi P_m [1 - \tau_B] = 0
\]

\[
\frac{\partial H}{\partial \alpha} = 0 \quad \Rightarrow [F(K_f) - C + M] \cdot (\lambda - \mu) = 0
\]

\[
\frac{\partial H}{\partial L} = 0 \quad \Rightarrow \eta + \varphi = 0
\]
Analysis of the Results

The economic interpretation of the optimality conditions is fairly straightforward:

Equation (13) implies that the marginal benefit of consumption (i.e., marginal utility) equals its opportunity cost \([a\lambda + (1 - \alpha) \mu]\), which is the marginal value of investment foregone. The value of marginal investment is a weighted average of the value of capital in both production sectors. As will become apparent, the value of a unit of capital in both sectors is equal, thus the implication is that the marginal utility of consumption equals the value of a unit of capital which is foregone.

Equation (14) states that the value of a unit of investment \([a\lambda + (1 - \alpha) \mu]\) (afforded by one additional unit of imports of capital goods) equals its opportunity cost. The latter is the amount of foreign exchange (withdrawn from reserve stocks), adjusted for the deterioration in credit terms caused by the decline in import coverage by reserves. We note that if imports had no effect on the term of credit (i.e., \(r_2 = 0\)) then the value of investment equals the shadow price of foreign exchange needed to implement that investment.

Equation (15) implies that value of capital in each of the two production sectors (that is, the export sector and the investment-consumption sector) is equal, (i.e., \(\lambda = \mu\)) even though capital is not transferable between the sectors once investments are implemented (except in the case \(F(K) - C + M = 0\), which implies no investment in the economy). Given \(\lambda = \mu\), it follows that the value of capital is higher than the value of the direct foreign exchange cost of a unit of capital imports. This is explained (as indicated above) by the additional cost, in terms of reduced credit worthiness, of imports.

Equation (16) implies that the value of a unit of reserves (or the shadow cost of foreign exchange) equals its opportunity cost (the
reduction of external debt). The value of a unit debt ($\eta$) is obviously negative, as the value of reserves ($\varphi$) is positive.

Using the result $\lambda = \mu = \varphi P_m (1 - \tau_2 B)$ (which also implies $\dot{\lambda} = \dot{\mu}$), equations (17) and (18) can be combined to yield

$$\frac{P_m F'}{P_g G'} = \left( \frac{1 - \tau_1 B}{1 - \tau_2 B} \right).$$

Equation (21) implies that in general the value of marginal product of capital in the two production sectors (evaluated at international prices) will not be identical. Obviously, in the case where no credit worthiness considerations are taken into account (i.e., $\tau_1 = \tau_2 = 0$), the marginal productivity of capital would be maintained equal in all sectors. Indeed, in an economy where most investments are made by private (and competitive) investors, one would expect them to operate such that $P_m F' = P_g G'$, since each one of them is too small to consider the national credit-worthiness implications of imports and exports. The planner (or the government), however, cannot ignore the impact of such variables as represented by the right hand side of equation (21). Thus, intervention in the sectoral allocation of investment is called for by export promotion and/or import substitution. Export promotion is reflected in the term $-\tau_1 B$ which measures (in units of output) the amount of subsidy that should be given to producers of the export good. Similarly, $-\tau_2 B$ measures (in physical units) the subsidy due to producers of the $F$ good (which can substitute imports of capital goods). Private investors will then equate (at the margin) the actual rates of return received by them, namely

$$P_g G' \cdot [1 - \tau_1 B] = P_m F' \cdot [1 - \tau_2 B],$$

which will ensure that the economy maintains the optimal condition given by equation (21). At the steady state, however, the optimal condition for the $F$ sector requires (from equation (17) with $\dot{\lambda} = 0$),

$$F' = \delta + \varphi.$$

Whether the export sector is (at the margin) more profitable than the import substituting sector depends on whether $\tau_1/\tau_2 \geq 1$. The size of $\tau_1/\tau_2$ is affected by the differential weight attached to the different credit-worthiness indicators and is thus an empirical question.

It should be noted that in practice, many governments find it more convenient to control the volume of imports (by tariffs or quotas), so as
to avoid foreign exchange crises which will deteriorate their credit status. In terms of the present model, the optimal tariff on imports is $-\tau_2 B$. Imposing such an import tax will eliminate the need to subsidize the $F$ sector.

Using the equality $\eta = -\varphi$, which implies $\varphi/\varphi = \eta/\eta$, equations (19) and (20) can be combined to yield

$$r - r_1 (\theta + \tau) B = i - r_2 B.$$  \tag{24}

The left hand side of equation (24) is the marginal cost of borrowing to the economy, which is necessarily higher than the rate of interest $r$, since it includes the extra cost due to the deterioration in terms of credit brought about by higher debt (the latter effect is represented by the term $-\tau_1 \cdot (\theta + \tau) \cdot B$). The right hand side represents the marginal benefit from reserve holdings, which is composed of the direct rate of interest plus the contribution to credit worthiness gained by additional reserves (given by $-\tau_2 B$). Equation (24) thus describes the optimality relation for reserve and debt stocks. Obviously, in the present model, if reserves had no impact on credit worthiness ($\tau_2 = 0$), then a zero reserves level should be maintained. This can be seen by observing that with $\tau_2 = 0$, it is impossible for equation (24) to hold (since $r - r_1 (\theta + \tau) B > r > i$ for all values of the arguments of $\tau$, as indicated in (4)). To approach the equality, $R$ will be depleted (and $B$ reduced by an equal amount), which verifies the assertion made above.

In an economy where part of the external borrowing is done by private corporations, the model points out the necessity of imposing a tax on foreign borrowing. The reason for such a tax is that while the cost of borrowing for each individual borrower is $r$ (the rate of interest) the cost to the economy is the marginal borrowing cost which includes the terms of credit effect. The presence of credit worthiness considerations thus implies an externality, which may be corrected by an appropriate tax. Obviously, direct measures can also be applied. For instance, recently, the government of Turkey banned foreign loans which are obtained at a rate of interest higher than a certain upper limit (Clarke, 1977). Other governments imposed administrative restrictions which amount to an implicit tax on borrowing from foreign sources.

Comparisons between the marginal rate of return and the external interest are possible, in the present model, only at the steady state. Thus, setting $\dot{\lambda} = \dot{\eta} = 0$ and using the results $\lambda = \mu$, $-\eta = \varphi$, one obtains from (17) and (19):
Equation (25) implies that at the steady-state, the marginal productivity of capital in the import-substituting sector (net of capital depreciation allowance) should be maintained at a level higher than the rate of interest of external debt. The difference is equal to the credit-worthiness effect of marginal borrowing. When credit-worthiness considerations are negligible (\( r_1 \approx 0 \)), the standard result is obtained, namely, the net marginal productivity of capital equals the interest rate.

The relation between the marginal rate of return of capital in the export sector and the rate of interest is less straightforward. Setting \( \dot{\mu} = \dot{\eta} = 0 \), and using the relation \( -\eta = \varphi = \mu/[P_m \cdot (1 - r_2 \cdot B)] \), one obtains from (18) and (19), at the steady state,

\[
\frac{P_g G'}{P_m} \cdot \frac{(1 - r_1 B)}{(1 - r_2 B)} = \delta = r - r_1 (\bar{\theta} + \dot{\theta}) \cdot B.
\]

Since the magnitude of the various terms on the left-hand side is not known a-priori, one cannot conclude as to whether the rate of return to capital should optimally be higher or lower than the direct rate of interest. We note, however, that the left-hand side describes the full impact of the marginal investment in the export sector: the direct contribution to export plus the improvement in credit terms induced by higher export \([P_g G' (1 - r_1 B)]\) discounted by the total cost of a unit of capital imports (that is, both direct and indirect cost). Equation (26) thus states that the full marginal contribution of investment in the export sector should equal the marginal cost of borrowing.

Setting \( \dot{\eta} = 0 \) also implies (by equation (19)) that at the steady state the rate of time preference of the economy (\( \varphi \)) exceeds the interest rate on external debt, as it needs to equal the marginal cost of borrowing.

From equation (17) with \( \dot{\lambda} = 0 \), one obtains the familiar "golden rule" of neo-classical models, asserting that at the steady state capital accumulation is maintained at a level such that the net marginal productivity equals the rate of time preference (if labor growth is nil). This rule applies, in the present model, only to the import-substituting sector. For the export sector the optimal steady state accumulation law is more complicated and depends on credit-worthiness effects.
Conclusion

By way of summary, the model implies the following set of policies for an economy with private borrowing, investment and production:

(i) Export promotion through a subsidy (of magnitude \(-P_g r_1 B\)) to the export sector;

(ii) Tax on capital goods imports (of magnitude \(-P_m r_2 B\)), which amounts to a policy of import substitution;

(iii) Tax on foreign borrowing by private corporations (of magnitude \(r_1 (\theta + \bar{\nu}) B\)).

These recommendations are, of course, of a partial nature, due to the simplistic structure of the model. But the paper makes the general argument that development policies should take into account the relationship between economic policies and the supply of foreign funds, and it demonstrates that such considerations require central government intervention in the operation of the market. This conclusion is most relevant for developing economies who depend on commercial international capital markets for a substantial portion of their foreign exchange inflow.

References

Borrowing, Capital Allocation and Credit-Worthiness Control


Zusammenfassung

Optimale internationale Schuldenaufnahme
Kapitalallokation und Kreditwürdigkeitskontrolle

Summary

Optimal International Borrowing, Capital Allocation and Credit-Worthiness Control

The paper develops a two-sector model of a growing economy, incorporating the relation between debt servicing capacity and the terms of credit facing the country. These terms are affected by macro-economic variables such as the volume of imports and exports, the size of foreign exchange reserves and the magnitude of debt service payments due on outstanding debt. The fact that importers, exporters and producers may be too small to take account of the impact their actions have on the economy's terms of credit creates discrepancies between private and public optimal solutions. The model suggests that optimal growth may require export promotion while simultaneously taxing both private borrowing of foreign funds and importation of capital goods.

Résumé

Endettement international optimal, allocation de capital et contrôle de la solvabilité

La présente étude développe un modèle bisectoriel d'économie en expansion, incluant la relation entre la capacité d'endettement et les conditions de crédit d'un pays. Ces conditions sont influencées par des variables macro-économiques telles que le volume des importations et des exportations, le stock des réserves de devises, le montant des paiements du service de la dette correspondant à l'endettement en cours. Le fait que les importateurs, les exportateurs et les producteurs n'ont fréquemment pas la taille requise pour influer par leur manière d'agir sur les conditions du crédit de l'économie provoque des disparités entre les endettements optimaux privé et public. Le modèle conclut que la croissance optimale requiert la promotion des exportations et parallèlement la taxation des emprunts privés en devises étrangères et de l'importation de biens d'investissement.
A Model for Analyzing Lenders' Perceived Risk

Optimal International Borrowing, Capital Allocation, and Credit Worthiness Control
Bank lending involves, in most cases, the risk that the borrower will not be able or willing to honour his obligations. The existence of default risk is an important factor in explaining the observed behaviour of lenders as demonstrated in the works of Jaffee and Modigliani (1969), Smith (1972), Azzi and Cox (1976), Jaffee and Rus-sel (1976), and others. Lenders' behaviour in this case depends crucially on their subjective evaluation of the probability of default. Thus, to explain lending behaviour, knowledge of lenders' subjective information is generally unobservable; and empirical analysis of lending behaviour is, therefore, difficult. Furthermore, there is often reason to believe that subjective information may vary considerably from lender to lender or from transaction to transaction because of previous experience, personal relationships, etc.; and, hence, the rôle of subjective perceptions cannot be ignored.

The purpose of this paper is to develop a model which facilitates inference about lenders' subjective default probabilities. Several alternative approaches of constructing both point and interval estimates of subjective default probabilities are proposed under various assumptions about competition, loss rate distribution, risk aversion, and relative loan size.

The assumption which makes these approaches possible is that lending takes place with a fixed probability of default. That is, lending transactions are assumed to be of sufficiently negligible size relative to the borrower's scale of operations (e.g. lending to sovereign borrowers) so that the probability of default is not influenced by the lender's current decision, i.e. the interest rate on the loan does not affect default probability. This assumption differs from that employed in the theoretical papers cited above and, hence, removes complications associated with endogenizing the probability of default. Nevertheless, the simplification which leads to the empirical possibilities developed in this paper is plausible in many situations-particularly in international lending. For example, suppose the state of São Paulo, Brazil, borrows $100 million in the Eurodollar market with the guarantee of the

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1Giannini Foundation Paper No. 523. The views expressed here are the authors'; they do not represent those of the World Bank. Estimates used in the text are provided from Euromarket data and are meant solely to be illustrative. There is no presumption that these estimates are the actual probabilities nor that the Bank in any way concurs with these perceptions.
federal government of Brazil. The likelihood of default in this case would surely be affected only slightly by the lender’s decision since the overall debt of Brazil is $32 billion dollars (Newsweek, 1978).

I. THE MODEL

Consider a simple one-period model similar to the one presented by Smith (1972). The lender is assumed to have a given amount, $W$, of loanable funds, all or part of which can be lent to the particular borrower under consideration at interest rate $\hat{r}$. The interest rate is exogenously given if the lender operates within a competitive capital market, or it depends on the volume of loans granted (according to the borrower’s demand schedule) if the lender has some degree of monopoly power. Alternatively, part or all of the lender’s funds can be lent at interest $r$ which is the opportunity cost of capital and which is considered risk free.² The borrower’s demand schedule is given by the relation,

$$f = f(L), \quad r' < 0 \text{ in the monopolist’s case}$$
$$r' = 0 \text{ for the competitive lender,} \quad (1)$$

where $L$ denotes the volume of loans granted. It will be useful to define the risk premium, $r$, which is the difference between the lender’s opportunity cost of capital and the interest rate actually paid by the borrower, namely,

$$\hat{r} \equiv r + \rho. \quad (2)$$

Using Equation 1 and considering $\rho$ as a constant, the demand schedule can be defined for the risk premium $r$,

$$r = r(L), \quad r' = r' \leq 0, \quad (3)$$

where, as before, $r' = 0$ applies to the case of the competitive lender.

The borrower is assumed to possess (at the time of loan negotiations) assets totalling $K$ which were financed in part by previous borrowing and in part from the borrower’s own resources. His total outstanding debt as of the time of the loan under consideration is given by $D$. Following Smith (1972), it is assumed that the liquidity situation of the borrower at the time of loan maturity is related to borrower resources (including the new loan) by a stochastic, linear homogenous function of the form $\gamma \times [K + L]$, where $\gamma$ is a continuous random variable with probability density function $f(\gamma)$ defined over the interval $(0, \infty)$. The parameters of $f$ are influenced by various borrower characteristics as viewed by the lender; and $f(\gamma)$ reflects, therefore, the latter’s subjective assessment of borrower prospects.

When the loan under consideration becomes due, the borrower must pay $(1 + \hat{r}) \times L$ in addition to other obligations which will be due on his older outstanding debt $D$, say $\phi \times D$ where $\phi$ denotes the rate of interest and amortization payments; $\phi$ is not affected by the new lender.

²As in the earlier works on lending under uncertainty cited above, portfolio considerations are ignored for the sake of simplicity.
A model for analysing lenders’ perceived risk

Obviously, if $\gamma$ will be sufficiently high so as to maintain the inequality

$$\gamma \times (K + L) - (1 + \rho) \times L - \phi \times D \geq 0,$$

(4)

then all creditors will be paid in full and no default takes place. Using Equation 4, one can define the critical $\gamma$ value, (say, $\gamma^*$)

$$\gamma^* = \frac{(1 + \rho) \times L + \phi \times D}{K + L}.$$

(5)

If $\gamma < \gamma^*$, not all obligations can be met, and a partial default takes place. In such a case, it is assumed that creditors are paid according to the proportion of debt they hold. In particular, the lender under consideration will be paid

$$\gamma \times (K + L) \times \frac{L}{(L + D)}.$$

The present value of the new lender’s wealth (say, $\gamma$) is thus given by

$$\gamma = \begin{cases} 
W - \frac{(1 + \rho)}{1 + \rho} \times L & \text{if } \gamma \geq \gamma^* \\
W - \gamma \times \frac{(K + L)}{(1 + \rho)} \times \frac{L}{(L + D)} & \text{if } \gamma < \gamma^*. 
\end{cases}$$

(6a)

Rearranging the terms in Equation 6, present value $\gamma$ can be rewritten as

$$\gamma = \begin{cases} 
W + r \times \theta \times L & \text{if } \gamma \geq \gamma^* \\
W - h \times L & \text{if } \gamma < \gamma^*. 
\end{cases}$$

(6b)

where

$$\theta \equiv \frac{1}{1 + \rho},$$

and

$$h \equiv 1 - \gamma \times \frac{K + L}{(L + D) \times (1 + \rho)}.$$

and

$$r = \text{risk premium as defined in Equation 2.}$$

Note that $h$ is random since it is a transformation of the random variable $\gamma$. From the formulation in Equation 7, it follows that, for a given value of $\gamma$, $h$ is the rate of loss incurred by the lender. The range of values over which $h$ is defined is $(0, \gamma^*)$. Thus, using the definition of $\gamma^*$ as given in Equation 5, $h$ is confined between upper and lower bounds $l$ and $h$, respectively, where

$$h \equiv \frac{\left\{ \frac{1 - \phi}{(1 + \rho)} \right\} \times D - \left\{ \frac{r}{(1 + \rho)} \right\} \times L}{(L + D)}.$$

$^3$It is implicitly assumed that no priority debt exists. However, one could assume that priority debt is paid first and that the other creditors share the remaining liquidity according to their share in regular debt. The appropriate analysis, however, remains unchanged.
Denote the lender's utility function by $U$, where $U$ is defined over present value of wealth. The utility function is assumed to have the properties

$$U' > 0, \quad U'' \leq 0,$$  
(7)

where $U'' < 0$ implies risk aversion and $U'' = 0$ reflects risk neutrality. It is further assumed that, in the case $U'' < 0$, relative risk aversion is no greater than one, i.e.,

$$-(U''/U') \times y \leq 1.$$

This is a plausible assumption as argued by Arrow (1971, p. 98). The lender's objective is to maximize the expected utility of present-value wealth by an optimal choice of loan size, i.e. using Equation 6b

$$\text{Max } \Pi = \int_{\gamma^*} \infty U(W + r \times \theta \times L) \times f(\gamma) \, d\gamma + \int_{0}^{\gamma^*} U(W - h \times L) f(\gamma) \, d\gamma.$$  
(8)

Note, however, that

$$\int_{\gamma^*} \infty U(W + r \times \theta \times L) f(\gamma) \, d\gamma = U(W + r \times \theta \times L) \times [1 - P(\gamma*)],$$

where

$$P(\gamma*) \equiv \int_{0}^{\gamma^*} f(\gamma) \, d\gamma$$

represents the probability of default. Note, also, using the definitions of $h$ and $h^*$, that

$$\int_{0}^{\gamma^*} U(W - h \times L) f(\gamma) \, d\gamma \equiv \int_{h}^{1} U(W - h \times L) \times p(h) \, dh$$

where

$$p(h) \equiv f \left[ (1 - h)(1 + \rho) \times \frac{(L + D)}{(K + L)} \right].$$

The objective function in Equation 8 can thus be written as

$$\text{Max } \Pi = [1 - P(\gamma*)] \times U(W + r \times \theta \times L) + \int_{h}^{1} U(W - hL) \times p(h) \, dh.$$  
(9)

In the general case, the probability of default $P(\gamma*)$, as well as the specific loss rate density $p(h)$, depends on the size of loan granted by the current lender and on the rate of interest charged for the loan (as inspections of $\gamma^*$ and $h$ verify). However, as will be shown below, in the case where the relevant loan size is small, relative to the size of the borrower's assets (i.e. when $L/K$ is small), the probability of default and the distribution of loss rates can be treated as given (from the point of view of the current lender). This simplifies the analysis considerably and facilitates several applications as will be demonstrated in the following sections.

In order to verify the assertions regarding $P(\gamma*)$ and $p(h)$, define the loan to asset ratio $\bar{L}$ (i.e. $\bar{L} \equiv L/K$). Rewriting the definitions of $\gamma^*$, $h$, and $h^*$ by simply dividing numerators and
A model for analysing lenders' perceived risk

denominators by $K$, one obtains

$$\gamma^* = \frac{(1 + \hat{r}) \times I + \phi \times (D/K)}{1 + \hat{I}}$$

$$h = 1 - \gamma \times \frac{(1 + \hat{I})}{[1 + (D/K)](1 + \rho)}$$

and

$$h = \left\{ \left[ \frac{1 - \phi}{(1 + \rho)} \right] \times \left( \frac{D}{K} \right) - \frac{[r \times \hat{I}/(1 + \rho)]}{[I + (D/K)]} \right\}.$$  

Taking the limit as $\hat{I} \to 0$ yields

$$\gamma^* \bigg|_{\hat{I}=0} = \phi \times \frac{D}{K};$$

$$h \bigg|_{\hat{I}=0} = \frac{1 - \gamma}{[(1 + \rho) \times (D/K)]};$$

$$h \bigg|_{\hat{I}=0} = \frac{1 - \phi}{1 + \rho}.$$  

Thus, when $I$ is sufficiently small, $\gamma^*$ and $h$ approach fixed parameters from the current lender's point of view, and the (random) loss rate $h$ is independent of the size of the current loan ($L$) and of the risk premium ($r$) charged on the loan.

These results imply that, with the assumption of negligible $\hat{I}$, the probability of default $P(\gamma^*)$ and the density of the loss rate $p(h)$ are not appreciably affected by the actions of the current lender and can be treated as fixed parameters in the lender's problem. Rather, the dominant factors underlying these probabilities are the existing economic prospects facing the borrower.

Before proceeding to characterize the optimal solution, one further modification needs to be made—namely, extending the model to a loan duration of $N$ years. Assuming that, at most, one default takes place within a loan’s duration and that loans are repaid in ‘balloon’ form (interest payments paid annually and the principal repaid in the last year), the formulation of the objective function (Equation 9) can be retained provided that $\theta$ is redefined as

$$\theta \equiv [1 - (1 + \rho)^{-N}] / \rho$$

(note that one can show that

$$\Sigma_{t=1}^{N} \hat{r} \times L \times (1 + \rho)^{-t} + L \times (1 + \rho)^{-N} - L = r \times \theta \times L),$$

and $h$ is reinterpreted as a discounted rate of loss. This is possible since both $\rho$ and $N$ are viewed as exogenous by the lender with the former parameter determined by market opportunities and the latter being dictated to a large extent by the lender's and borrower's overall liquidity projections (Beim, 1977, p. 723).

For notational convenience in the following analysis, define

$$P = P(\gamma^*), y_1 = W + r \times \theta \times L,$$ and $$y_n = W - h \times L.$$
Differentiation (Equation 9) with respect to \( L \), the following first-order condition characterizes the optimal solution:

\[
\frac{\partial \Pi}{\partial L} = (1 - P) \times \theta \times U'(y_1) \times (r + Lr') - \int_0^1 U'(y_h) \times h \times p(h) \, dh = 0. \tag{10}
\]

The second-order condition requires

\[
\frac{\partial^2 \Pi}{\partial L^2} = (1 - P) \times U''(y_1) \theta^2 (r + Lr')^2 + \int_0^1 U''(y_h) h^2 p(h) \, dh
+ (1 - P) \times \theta \times U'(y_1) \times (2r' + Lr'') < 0. \tag{11}
\]

From Equation 11, it is obvious that risk aversion is a necessary and sufficient condition for

\[
\frac{\partial^2 \Pi}{\partial L^2} < 0
\]

in the case of a competitive lender \((r' = r'' = 0)\). In the case of a risk neutral monopolistic lender,

\[
(2r' + Lr'') < 0
\]

is necessary and sufficient to guarantee concavity of the objective function. With risk aversion and monopoly, the latter condition is not necessary but remains sufficient. Henceforth, it will be assumed that second-order conditions hold for the lender under consideration.

II. COMPARATIVE STATIC RESULTS

Before proceeding to demonstrate the wide range of empirical uses of this model, it is first useful to examine the various comparative static properties of the model for the purpose of showing that the model is indeed plausible.

Demand elasticity

Considering the monopolistic lender, Equation 10 implies that the lender operates on the nonelastic portion of the borrower’s demand schedule. To see this, note that \( r + Lr' = r \times (1 - \eta) \), where \( \eta \) is the absolute value of demand elasticity at the optimal point. Obviously, if \( \eta > 1 \), Equation 10 cannot hold since the left-hand side is negative.

Default probability

A borrower who is more risky (i.e., who carries a higher probability of default) will be granted less credit whether the lender is competitive or not. This can be confirmed by assuming that, for any given rate of loss \( h \), the relative likelihood is at least as great as before, \( d \times p(h) > 0 \) for \( h \leq h \leq 1 \).

Hence, using the fact that

\[
P = \int_0^1 p(h) \, dh,
\]
A model for analysing lenders' perceived risk

one finds that

$$\int_{h}^{1} dp(h) > 0$$

if $dP > 0$. By differentiating Equation 10, one then obtains

$$\frac{dL}{dP} = \left[ \frac{a^2 L^2}{\partial L^2} \right]^{-1} \times \int_{h}^{1} \left[ U'(y_1) \times \theta \times (r + Lr') + U'(y_h) \times h \right] \frac{dp(h)}{dp} < 0,$$

where the sign is established using Equation 11 and the earlier result that $r + Lr' > 0$ for both a monopolist and a competitive lender. The result in Equation 12 implies, in the case of a monopolist, that riskier borrowers are charged a higher risk premium. This is also the case in a competitive market since, with a higher probability of default, a smaller amount of loans will be offered by any individual lender in the market at any given interest rate. The aggregate supply of loans to the borrower under consideration will thus decline. With a negatively sloped demand for loans and a positively sloped aggregate supply, a decline in supply must yield a higher equilibrium level of risk premium (with a lower amount of loans in a competitive market is indeed positively sloped (i.e., that supply is increasing with higher interest rates). This is done by differentiating Equation 10:

$$\frac{dL}{dr} = - \left( \frac{a^2 L^2}{\partial L^2} \right)^{-1} (1 - P) \times \theta \times U'(y_1) \times \left[ 1 + \frac{U''(y_1)}{U'(y_1)} rL \right].$$

Under the assumption that relative risk aversion is no greater than one, the term in square brackets on the right-hand side of Equation 13 is positive; and it is thus concluded that

$$dL/dr > 0.$$

It should be noted in the case where the lender considers the loan to have an impact on the probability of default, however, that the offer curve by each individual lender is backward bending irrespective of attitudes toward risk as shown in Jaffee and Modigliani (1969), Smith (1972), and Azzi and Cox (1976). The different results in the present analysis are due to the fact that a higher interest rate increases the marginal expected utility while, in the earlier models, a higher interest rate may reduce expected utility by causing a higher probability of default.

Risk aversion

Considering the role of risk aversion in the model, it is intuitively expected that, when lenders are more risk averse, the volume of loans will be lower and risk premiums will be higher. This can be shown simply by assuming a specific form of the utility function such that risk aversion is reflected in a single parameter. The two most common such utility functions are the constant relative risk-aversion family of functions and the constant absolute risk-aversion family of functions. These are given, respectively, by

$$U(y) = ay^{1 - \alpha}, \ 0 < \alpha < 1$$

and

$$U(y) = a - e^{-\alpha y}, \ \alpha > 0$$

(14a) (14b)
where $\alpha$ is a parameter of risk aversion such that the higher $\alpha$ is associated with higher risk aversion. Assuming the case of Equation 14a, differentiation of Equation 10 obtains

\[
\frac{dL}{d\alpha} = \left[ \frac{\partial^2 \Pi}{\partial L^2} \right]^{-1} \times (W + r\theta L)^{-\alpha} \times \left[ (1 - P) \times \theta \times (r + Lr') \times \int_h^1 h \times (W - hL)^{-\alpha} \times p(h) \ln \left( \frac{W - hL}{W + r\theta L} \right) \times dh \right].
\]

Since

\[
\ln \left( \frac{(W - hL)}{(W + r\theta L)} \right) < 0,
\]

it follows that

\[
\frac{dL}{d\alpha} < 0.
\]

Similarly, assuming the case of Equation 14b and differentiating Equation 10 yields

\[
\frac{dL}{d\alpha} = \left[ \frac{\partial^2 \Pi}{\partial L^2} \right]^{-1} \times \left[ \int_h^1 h \times p(h) \times e^{-\alpha(W - hL)} \times (h + r\theta) \times L \times dh \right] < 0.
\]

The results in Equations 15a and b imply that a higher degree of risk aversion causes a smaller volume of loans and higher risk premiums in both competitive and noncompetitive markets.

**III. SUGGESTED APPLICATIONS**

The results in the previous section are plausible and intuitive and imply that the equilibrium relation in Equation 10 can be usefully considered for further expositional and empirical purposes. In this section several such empirical applications are demonstrated. For this purpose, suppose that lenders' information and evaluation distinguish between the overall probability that a default will take place and the (conditional) probability that a given rate of loss ($h$) will be incurred. Hence, for conceptual purposes, and so that subjective default probability can be meaningfully discussed, suppose that the probability density function of loss rate $h$ is proportional to the overall default probability such that the conditional probability of $h$ depends on $h$ alone and not on the economic factors which explain the overall probability. This implies

\[
\frac{p(h)}{P} = \psi(h), \quad \int_h^1 \psi(h) \, dh = 1,
\]

where $\psi$ depends only on $h$ irrespective of the economic factors which determine $P$. Note, that, with this assumption, the (unconditional) probability of any given loss rate increases when the overall default probability increases, as one would expect, since Equation 16 implies

\[
p(h) = \psi(h) \times P;
\]
A model for analysing lenders' perceived risk

thus,

\[
\frac{dP(h)}{dP} = \psi(h) > 0.
\]

Applying Equation 16 in Equation 10 and rearranging yields

\[
r = \frac{1}{(1 - \eta)} \frac{P}{(1 - P) \theta} R
\]

where

\[
\bar{h} \equiv \int_{h}^{1} \psi(h) h \, dh
\]

and

\[
R = \frac{\int_{h}^{1} h \times \psi(h) \times U'(y_h) \, dh}{\bar{h} \times U'(y_1)}.
\]

For a monopolistic lender, Equation 17 demonstrates that the risk premium includes a demand elasticity factor, \((1 - \eta)^{-1}\); the odds of default, \(P/(1 - P)\); a time-effect factor, \(\theta^{-1}\); an average loss-rate factor \(\bar{h}\); and a risk-aversion factor, \(R\). The latter factor is identically equal to one under risk neutrality in which case

\[U'(y_h) = U'(y_1).\]

With risk aversion, it is easy to show that

\[R > 1 \text{ and } \partial R/\partial L > 0.\]

Equation 17 offers a great deal of flexibility in empirical inference of subjective default probabilities because of its simplicity. This flexibility is demonstrated in the following three applications.

Threshold default probability

A simple application of the model is related to the concept of 'threshold probability' for competitive lenders. The threshold probability is the highest value of default probability (say, \(P^\star\)) for which a loan will be granted to any borrower and depends on \(r, \theta, \text{ and } \bar{h}\). Borrowers with probability lower than \(P^\star\) will be given credit with the amount of credit increasing in the ratio

\[
\frac{[P^\star/(1 - P^\star)]}{[P/(1 - P)]}.
\]

Using Equation 17, the threshold probability is calculated by noting that, at \(P = P^\star\), it must hold that \(L = 0\) which implies that

\[U'(y_1) = U'(y_h) = U'(W).\]
This obtains

\[ P^* = \frac{r}{(h/\theta) + r}. \]  

(18)

By differentiating Equation 18, one can verify that the threshold probability is positively related to the risk premium and the loan duration (i.e.

\[ \frac{\partial P^*}{\partial r} > 0, \frac{\partial P^*}{\partial N} > 0 \]

while being negatively related to the average loss rate and to the opportunity cost of capital (i.e.

\[ \frac{\partial P^*}{\partial h} < 0, \frac{\partial P^*}{\partial \rho} < 0. \]

Using Equation 18 in 17, with \( \eta = 0 \), further yields

\[ R(L) = \frac{P^*}{(1 - P^*)} \frac{P}{1 - P}. \]

(19)

The right-hand side of Equation 19 is the ratio of odds evaluated at the threshold and actual values of probability. From Equation 19, one can derive the result that loans (and the risk aversion premium) increase as the right-hand side of Equation 19 increases.

A useful property of the threshold probability is that it is independent of the degree of risk-aversion as is apparent from Equation 18. Since the values of \( \rho, N, \) and \( h \) are either known or can be estimated without major difficulty, one can calculate the threshold probability which applies to different risk premiums. This approach is demonstrated in Table 1 for \( \rho = .065 \) and \( h = .15, .2 \). The results are presented in terms of \( q^* \) (the short-run threshold probability) so as to allow comparison between cases with different loan durations. That is, if at most, one default can take place within the duration of the loan, then the probability \( P \) of default over the entire period of the loan is

\[ P = 1 - (1 - q)^N, \]

provided that \( q \), the probability of default in any given year, is constant.\(^4\) It is reasonable to expect that, given current data, lenders assume \( q \) to remain essentially constant since no data beyond the period in which the loan is granted are available for projection. The \( q^* \) in Table 1 is a threshold or upper bound on \( q \) and is related to \( P^* \) just as \( q \) is related to \( P \).

Using this approach, one can very easily confine the subjective probability \( q \) to the interval \([0, q^*]\) under competition by simply using data on \( r, \rho, N \) and \( h \) (note that \( \theta \) is determined by \( \rho \) and \( N \)); moreover, the interval \([0, P^*]\) is fairly narrow even for small \( h \). Furthermore, if \( h \) is unknown but can be bounded from below by \( \tilde{h}^* \), then \( P \) can be confined to the interval \([0, P^{**}]\) where

\[ P^{**} = \frac{r}{(\tilde{h}^*/\theta) + r} \]

(since \( P^* \) is decreasing in \( \tilde{h} \)) or \( q \) can be confined to an associated interval \([0, q^{**}]\) where \( P^{**} = 1 - (1 - q^{**})^N \). Hence, the simple transaction data on \( r, \rho, \) and \( N \) is sufficient.

\(^4\)The reader will note that this definition of \( P \) was indeed in the description of the model.
A model for analysing lenders' perceived risk

Table 1. Threshold short-run probabilities and risk premiums for 5- and 10-year loans

<table>
<thead>
<tr>
<th>Loan duration (years)</th>
<th>( r ) (per cent)</th>
<th>( q^* ) (( \bar{h} = .15 ))</th>
<th>( q^* ) (( \bar{h} = .2 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = 5 )</td>
<td>.25</td>
<td>.013</td>
<td>.010</td>
</tr>
<tr>
<td>( N = 10 )</td>
<td>.25</td>
<td>.011</td>
<td>.009</td>
</tr>
<tr>
<td>( N = 5 )</td>
<td>.50</td>
<td>.026</td>
<td>.020</td>
</tr>
<tr>
<td>( N = 10 )</td>
<td>.50</td>
<td>.021</td>
<td>.016</td>
</tr>
<tr>
<td>( N = 5 )</td>
<td>.75</td>
<td>.037</td>
<td>.029</td>
</tr>
<tr>
<td>( N = 10 )</td>
<td>.75</td>
<td>.03</td>
<td>.024</td>
</tr>
<tr>
<td>( N = 5 )</td>
<td>1.00</td>
<td>.048</td>
<td>.037</td>
</tr>
<tr>
<td>( N = 10 )</td>
<td>1.00</td>
<td>.038</td>
<td>.03</td>
</tr>
<tr>
<td>( N = 5 )</td>
<td>1.25</td>
<td>.058</td>
<td>.045</td>
</tr>
<tr>
<td>( N = 10 )</td>
<td>1.25</td>
<td>.046</td>
<td>.036</td>
</tr>
<tr>
<td>( N = 5 )</td>
<td>1.50</td>
<td>.067</td>
<td>.053</td>
</tr>
<tr>
<td>( N = 10 )</td>
<td>1.50</td>
<td>.053</td>
<td>.042</td>
</tr>
<tr>
<td>( N = 5 )</td>
<td>1.75</td>
<td>.076</td>
<td>.060</td>
</tr>
<tr>
<td>( N = 10 )</td>
<td>1.75</td>
<td>.059</td>
<td>.048</td>
</tr>
<tr>
<td>( N = 5 )</td>
<td>2.00</td>
<td>.085</td>
<td>.067</td>
</tr>
<tr>
<td>( N = 10 )</td>
<td>2.00</td>
<td>.065</td>
<td>.053</td>
</tr>
</tbody>
</table>

* Assuming the opportunity cost of capital \( \rho \) is 6.5 per cent and \( N = 5 \), the value of \( \theta \) is 4.156, with \( N = 10 \), the value of \( \theta \) is 7.1888.

To develop some information about the magnitudes of subjective default probabilities which might exist.

An application in inferring lenders' subjective probabilities: The case of the Eurodollar market

Another application of the model in this paper can be made by inferring interval estimates of lenders' subjective probabilities of default from a minimal amount of observed market data. To demonstrate this possibility, 20 observations on loans granted in the Eurodollar market during the third quarter of 1973 have been used. These observations (presented in the Appendix) refer to loans made during that period to public (or publicly guaranteed) entities in developing countries for which data on risk premium and loan duration were available.²

² The short period covered was chosen so as to avoid any significant variations in the opportunity cost of capital among observations. The estimates are meant solely to be illustrative. There is no presumption that these estimates are the actual (objective) probabilities nor that the World Bank in any way concurs with these perceptions.
The procedure employed here uses the assumptions that the market operates competitively and that the subjective loss-rate distribution is singular in order to establish bounds on the lenders' perceived default probabilities. The first assumption implies that \( \eta = 0 \), and the second implies that \( h \) is a constant, say, \( \bar{h} \). Assuming that the utility function can be specified by the constant elasticity formulation (Equation 14b), the risk-aversion premium \( R \) can then be written

\[
R = \left( \frac{1 + r \theta l}{1 - \bar{h}l} \right)^{\alpha},
\]

(20)

where \( l \equiv L/M \) is the share of the loan in total loanable funds and \( \alpha \) is the relative risk-aversion parameter \( (0 < \alpha < 1) \).

While the 20 observations on Eurodollar loans range in loan duration between 10 and 15 years, the probability concept which is of interest is again one that is free of the loan duration effect. Thus, combining Equations 17 and 20 and the relation between \( q \) and \( P \) the following formula is obtained:

\[
q = 1 - (1 - P)^{1/N} = 1 - \left[ 1 + \frac{\theta \bar{h}}{\bar{h}} \left( \frac{1 - \bar{h}l}{1 + \theta l} \right)^{\alpha} \right]^{-1/N}.
\]

(21)

To make use of Equation 21 in the Eurocurrency market, it can be observed that the average rate of loss in international banking is low; in almost all cases of debt-service problems, loans are renegotiated and rescheduled.\(^6\) Hence, for exemplary purposes, a value of \( \bar{h} = .1 \) is used.\(^7\) As for \( l, \rho, \) and \( \alpha \), it can be noted from Equation 21 that

\[
\frac{\partial q}{\partial l}, \frac{\partial q}{\partial \rho}, \text{ and } \frac{\partial q}{\partial \alpha} < 0.
\]

Thus, specifying sets of upper and lower bounds on these coefficients leads to lower and upper bounds, respectively, for the short-run probability \( q \). For example, consider upper bounds of \( l = .3, \rho = .08, \) and \( \alpha = .8 \) and lower bounds of \( l = 0, \rho = .06, \) and \( \alpha = 0 \). These values seem to span the range of reality. That is, \( l = 0 \) and \( \alpha = 0 \) are the lowest possible values (assuming no risk-loving behaviour), while \( l = .3 \) and \( \alpha = .8 \) seem to be above and beyond all likely possibilities. No bank seems to loan anything near 30 per cent of its funds to one borrower; also \( \alpha \) close to 1, in the case of the constant elasticity utility function, implies an extremely high degree of risk aversion. Finally, the opportunity cost of capital seems to be between 6 and 8 per cent. Using these bounds, the associated interval estimates for subjective default probability (as perceived by Eurodollar lenders) in Table 2 are possible for the Eurodollar market transactions considered above. However, these estimates do not necessarily imply judgments on the objective (true) probability.

Several interesting observations can be made on the basis of these results. First, the upper and lower bounds span a range for the probability of each case within .8 of 1 per cent or less. This is a small interval relative to the overall variation in Table 2. Second, the calculated \( q \) s for those countries with more than one observation (e.g., Algeria) are fairly close for most

\(^6\) In this respect, Friedman (1977, p. 55) noted: ‘Losses in U.S. banks’ overseas operations have been less, both in absolute terms and as a proportion of total risk assets, than in U.S. operations.’ A similar observation has been made by Beim (1977, p. 717).

\(^7\) Normally, one would want to estimate \( \bar{h} \) since the estimates are somewhat sensitive to this particular parameter.
A model for analysing lenders' perceived risk

Table 2. Perceived short-run probabilities in the Euromarket third quarter, 1973

<table>
<thead>
<tr>
<th>Observation number</th>
<th>Country</th>
<th>q upper bound&lt;sup&gt;a&lt;/sup&gt;</th>
<th>q lower bound&lt;sup&gt;b&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Algeria (N = 15)</td>
<td>.044</td>
<td>.039</td>
</tr>
<tr>
<td>2</td>
<td>Algeria (N = 12)</td>
<td>.047</td>
<td>.042</td>
</tr>
<tr>
<td>3</td>
<td>Algeria (N = 10)</td>
<td>.048</td>
<td>.044</td>
</tr>
<tr>
<td>4</td>
<td>Brazil (N = 10)</td>
<td>.052</td>
<td>.049</td>
</tr>
<tr>
<td>5</td>
<td>Brazil (N = 12)</td>
<td>.049</td>
<td>.044</td>
</tr>
<tr>
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<td>Colombia</td>
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<td>.039</td>
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<tr>
<td>7</td>
<td>Gabon</td>
<td>.079</td>
<td>.072</td>
</tr>
<tr>
<td>8</td>
<td>Greece</td>
<td>.040</td>
<td>.036</td>
</tr>
<tr>
<td>9</td>
<td>Iran (N = 12)</td>
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<td>.033</td>
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<td>.033</td>
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<td>Korea (N = 10)</td>
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<td>.065</td>
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<td>15</td>
<td>Nicaragua</td>
<td>.066</td>
<td>.060</td>
</tr>
<tr>
<td>16</td>
<td>Peru</td>
<td>.075</td>
<td>.068</td>
</tr>
<tr>
<td>17</td>
<td>Philippines</td>
<td>.087</td>
<td>.079</td>
</tr>
<tr>
<td>18</td>
<td>Senegal</td>
<td>.083</td>
<td>.075</td>
</tr>
<tr>
<td>19</td>
<td>Zaïre</td>
<td>.079</td>
<td>.072</td>
</tr>
<tr>
<td>20</td>
<td>Zambia</td>
<td>.069</td>
<td>.063</td>
</tr>
</tbody>
</table>

<sup>a</sup> Calculated with α = 0, l = 0, ρ = .06, and h = .1.
<sup>b</sup> Calculated with α = .8, l = .3, ρ = .08, and h = .1.

of these cases (theoretically, they should be identical). Also, Peru, Zaïre, and Zambia, which seem to have been considered as higher risks by Euromarket lenders as indicated by their q value, have indeed experienced serious debt-service problems in years after 1973. These observations may serve as an indication of the creditability of the assumptions. In summary, it appears that fairly precise estimates of default probabilities, as perceived by lenders, are possible under competition using only the terms of the loan if loss rate for the event of default can be determined (estimated).

Possible econometric applications

In addition to the above examples, Equation 17 also has interesting econometric possibilities. For the purpose of econometric work, it is useful to consider the common logistic specification for the probability, namely,

\[ P = \frac{e^{\beta_0 + \beta'X}}{1 + e^{\beta_0 + \beta'X}} \]

(22)
where \( \beta_0 \) is a constant, \( \beta \) is a vector of coefficients, and \( X \) is a vector of economic indicators including loan duration which are considered by lenders as relevant risk indicators. With this specification, the odds are log linear, i.e.,

\[
\ln \left( \frac{P}{1 - P} \right) = \beta_0 + \beta'X.
\]

Hence, Equation 17 becomes

\[
\ln r = -\ln (1 - \eta) + \ln \bar{h} - \ln \theta + \beta_0 + \beta'X + \epsilon,
\]

where

\[ \epsilon \equiv \ln R \equiv \ln \left[ \int_{h}^{1} h \psi(h)U'(y_{1}) \, dh \right] - \ln \{\bar{h}U'(y_{1})\}. \]

The model in Equation 23 has several interesting potential simplifications. First, for risk-neutral lenders, it is clearly the case from above that \( \epsilon = \ln R = 0 \); hence,

\[
\ln r = -\ln (1 - \eta) + \ln \bar{h} - \ln \theta + \beta_0 + \beta'X.
\]

With the presence of a disturbance term, Equation 23 can be estimated in a variety of situations to determine which factors are considered important by lenders in assessing default probabilities. For instance, \( \eta, \bar{h}, \) and \( \theta \) can be calculated (\( \theta \) depends only on \( \rho \) and \( N \) which are often recorded, and \( \bar{h} \) is also occasionally recorded for different classes of borrowers) and then included in a regression estimating \( \beta_0 \) and \( \beta \) where a coefficient of 1 or \(-1\) is imposed for \( \ln (1 - \eta) \), \( \ln \bar{h} \), and \( \ln \theta \) as needed according to Equation 24. Alternatively, one may find data where combinations of \( \eta, \bar{h}, \) and \( \theta \) are held constant and can thus be included in a constant term. Another approach (Feder and Just, 1977a) is to consider some of the first right-hand terms as randomly distributed among borrowers and then use a variance-components approach in estimating \( \beta \) from a time series of cross-section data.

Finally, an interesting observation can be made for Equation 23 in the general case of risk preferences. As it stands, the \( \epsilon \) term in Equation 23 would be very difficult to treat econometrically: besides, it would imply inclusion of the dependent variable \( r \) on the right-hand side. But in a broad range of cases, it turns out that \( \epsilon \) is negligible relative to both \( \ln r \) and the usual variations in \( \ln r \) among observations. This can be demonstrated assuming a constant elasticity utility function in which case Equation 20 applies. Using the same plausible limits on parameters used to generate Table 2, together with additional limits on \( r \) of \( 0 \leq r \leq 0.025 \) and on \( N \) of \( 0 \leq N \leq 10 \), it can be shown through simple calculations that \( 1 \leq R \leq 1.07 \) and hence \( 0 < \ln R < 0.068 \). By comparison, as \( r \) varies from 0.00575 to 0.025 (which seems to span the observed range of premiums in the Eurodollar market), the dependent variable \( \ln r \) varies from \(-3.698\) to \(-5.159\). Hence, even with most conceivable limits on \( l, \rho, \) and \( \alpha \), it is thus clear that \( \epsilon \) is negligible and can, for most practical purposes, be included with any disturbance in Equation 23.

Considering a competitive market, Equation 24 can become fairly simple once \( \epsilon \) is incorporated in the error term if competition implies \( \eta = 0 \). The equation is then reduced to

\[
\ln r = \ln \bar{h} - \ln \theta + \beta_0 + \beta'X + \nu
\]
where $v$ is a random disturbance. This formulation is exemplified in the following estimation example which pertains to transactions in the Eurodollar market in the period January, 1975, using a set of 29 observations on public or publicly guaranteed loan transactions (corresponding to 23 developing countries) as reported in Appendix 2. Reliable indications in the period covered imply that the cost of capital ($\rho$) was quite stable. For instance, the Eurodollar three-month and six-month deposit rates (which are closely related to the cost of capital) ranged from 6 per cent to 7 per cent and 7 per cent to 8 per cent, respectively, while in the previous year they fluctuated substantially from 8.5 per cent to 14 per cent. Adopting the assumption that $\rho$ was approximately constant throughout the period covered thus simplifies the treatment of $\theta$ in Equation 25 since one can experiment with a number of likely values of $\rho$ (say, .06, .07, and .08) in alternative estimates. The vector $X$ includes, in addition to loan duration (which is expected to be positively related to default probability), three variables which have been suggested as potentially related to debt-servicing capacity of developing countries and, hence, to the risk of default (Frank and Cline, 1971, and Feder and Just, 1977b). These variables are:

1. The ratio of debt-service payments to exports (also referred to as the debt-service ratio). Higher ratios reflect a heavier burden of debt service payments and are thus expected to be positively related to the probability of default.
2. The ratio of imports to G.N.P., which reflects claims of foreign exchange which are competing with debt-service obligations. Higher import to G.N.P. ratios should thus be positively related to the probability of default.
3. The level of G.N.P. per capita as a proxy for degree of development and the ability to divert resources away from consumption in order to service external debt. Higher per capita G.N.P. is thus negatively related to default risk.

Data for the calculation of these variables were obtained from various World Bank and International Monetary Fund publications. Having defined the vector $X$ and using three alternative values for $\rho$ for the calculations of $\theta$, the corresponding three regression estimates of Equation 25 are reported in Table 3. The results are practically identical for the different values of $\rho$ and thus imply that lack of information regarding the exact cost of capital is not serious. All parameters are statistically significant at a 7.5 per cent one-sided significance level, and their signs confirm the a priori arguments advanced. Thus, longer loan maturities, higher debt-service ratios, and higher import to G.N.P. ratios increase the perceived probability of debt-service problems of borrowing governments while higher per capita incomes reduce the perceived probability.

To examine one useful application of results of this type, note that the constant term reported in Table 3 is an estimate of $\ln \bar{h} + \beta_0$ (plus the mean of the error term $v$ if the latter has a nonzero mean). For given values of $\bar{h}$ (and mean error when applicable), one can calculate the corresponding $\beta_0$ values. These, together with the $\beta$ coefficients reported in Table 3, can be used in calculating perceived probabilities using Equation 22, where each alternative value of $\beta_0$ corresponds to a set of probabilities (Feder and Just, 1977a).

Using the parameters in Table 3, one can also establish a direct link between terms of

---

6 This period provides a larger representation of countries in the sample than other periods, thus contributing to the quality of econometric results.

9 One-sided tests are more appropriate in the present case since there are strong a priori arguments relating to the signs of the coefficients.
credit and economic performance criteria which can be useful for a borrower in managing economic resources wisely. Consider, for example, a five-year loan sought by an economy with a G.N.P. of $1000 per capita. Table 4 presents the risk premiums corresponding to various debt-service ratio and import to G.N.P. ratio combinations. Using the results of this analysis, a borrowing country can thus evaluate the interaction of, say, import policy and borrowing policy with respect to its terms of credit. In a similar way, one can represent the risk premiums corresponding to other combinations of explanatory variables.

Table 3. Regression results with Eurodollar data: January–July, 1975

<table>
<thead>
<tr>
<th>Variable</th>
<th>Case 1 (p = .06)</th>
<th>Case 2 (p = .07)</th>
<th>Case 3 (p = .08)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan duration (years)</td>
<td>.153</td>
<td>.149</td>
<td>.145</td>
</tr>
<tr>
<td>(6.515)*</td>
<td>(6.341)</td>
<td>(6.171)</td>
<td></td>
</tr>
<tr>
<td>Debt-service ratio</td>
<td>.467</td>
<td>.467</td>
<td>.467</td>
</tr>
<tr>
<td>(1.671)</td>
<td>(1.672)</td>
<td>(1.673)</td>
<td></td>
</tr>
<tr>
<td>Import to G.N.P. ratio</td>
<td>.432</td>
<td>.431</td>
<td>.431</td>
</tr>
<tr>
<td>(2.511)</td>
<td>(2.510)</td>
<td>(2.510)</td>
<td></td>
</tr>
<tr>
<td>Per capita G.N.P. ($1000)</td>
<td>-.089</td>
<td>-.088</td>
<td>-.088</td>
</tr>
<tr>
<td>(1.541)</td>
<td>(1.541)</td>
<td>(1.541)</td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>-3.496</td>
<td>-3.502</td>
<td>-3.509</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>.69</td>
<td>.68</td>
<td>.67</td>
</tr>
<tr>
<td>( \bar{R}^2 )</td>
<td>.64</td>
<td>.63</td>
<td>.61</td>
</tr>
</tbody>
</table>

*Figures in parentheses are 't' ratios.

Table 4. Risk premiums for a five-year loan by a borrowing country with $1000 per capita G.N.P. *

<table>
<thead>
<tr>
<th>Debt-service ratio</th>
<th>Import to G.N.P. ratio</th>
<th>.1</th>
<th>.2</th>
<th>.3</th>
<th>.4</th>
<th>.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>.1</td>
<td>.0155</td>
<td>.0162</td>
<td>.0169</td>
<td>.0177</td>
<td>.0184</td>
<td></td>
</tr>
<tr>
<td>.2</td>
<td>.0163</td>
<td>.0170</td>
<td>.0177</td>
<td>.0185</td>
<td>.0193</td>
<td></td>
</tr>
<tr>
<td>.3</td>
<td>.0170</td>
<td>.0178</td>
<td>.0186</td>
<td>.0194</td>
<td>.0202</td>
<td></td>
</tr>
</tbody>
</table>

*The calculations are based on Case 2 of Table 3, but almost identical results are obtained with Cases 1 and 3.
IV. CONCLUSION

This paper presents a model of lending under default risk. With an underlying assumption that the probability of default is determined by factors not controlled by the lender (which is essentially the case when a new loan is small relative to the borrower's existing debt), the effects of risk, risk aversion, and expected loss rate on the equilibrium levels of loan supply and risk premiums are analysed for both competitive and noncompetitive markets. The model is shown to be useful for assessing subjective default probabilities from market data, using the Eurodollar market as an example. The model also enables the determination of upper bound threshold probabilities for acceptable customers, given the relevant duration of loan and interest rate. Further utilization of the model for various econometric purposes is discussed in detail, and it is shown that in many cases the complicated term reflecting risk aversion can be ignored since its magnitude and range of variation are negligible. This simplifies greatly the procedures for econometric application of the model as demonstrated in an example utilizing additional Eurodollar market data.

Each of these approaches can provide useful information depending on data availability. If one simply observes the risk premium, cost of capital, loan duration, and average loss rate in the event of default, then the lender's subjective probability can be confined to an interval $(0, \hat{P}^*)$. If, in addition, bounds can be established on the share of the individual loan of the lender's total loanable funds and on the lender's risk aversion, the subjective probabilities can be confined to a much narrower interval. Finally, if information can alternatively be gained on the economic factors which influence a lender's subjective probability, then regression techniques can be used to produce point estimates of lender's subjective probabilities under certain conditions. Results of the latter type show particular promise for a borrower in managing credit worthiness since they provide information on how the borrower's other economic decisions affect its cost of capital.

APPENDIX

Table 1A. Transactions in the Eurodollar market: third quarter, 1973

<table>
<thead>
<tr>
<th>Borrowing country</th>
<th>Risk premium (per cent)</th>
<th>Loan duration (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algeria</td>
<td>1.008</td>
<td>15</td>
</tr>
<tr>
<td>Algeria</td>
<td>.938</td>
<td>12</td>
</tr>
<tr>
<td>Algeria</td>
<td>.875</td>
<td>10</td>
</tr>
<tr>
<td>Brazil</td>
<td>1.00</td>
<td>12</td>
</tr>
<tr>
<td>Brazil</td>
<td>.955</td>
<td>10</td>
</tr>
<tr>
<td>Colombia</td>
<td>.75</td>
<td>10</td>
</tr>
<tr>
<td>Gabon</td>
<td>1.75</td>
<td>10</td>
</tr>
<tr>
<td>Greece</td>
<td>.75</td>
<td>12</td>
</tr>
<tr>
<td>Iran</td>
<td>.688</td>
<td>12</td>
</tr>
<tr>
<td>Borrowing country</td>
<td>Risk premium (per cent)</td>
<td>Loan duration (years)</td>
</tr>
<tr>
<td>-------------------</td>
<td>-------------------------</td>
<td>-----------------------</td>
</tr>
<tr>
<td>Iran</td>
<td>.625</td>
<td>10</td>
</tr>
<tr>
<td>Korea</td>
<td>1.205</td>
<td>11</td>
</tr>
<tr>
<td>Korea</td>
<td>1.5</td>
<td>10</td>
</tr>
<tr>
<td>Mexico</td>
<td>.659</td>
<td>12</td>
</tr>
<tr>
<td>Mexico</td>
<td>.575</td>
<td>10</td>
</tr>
<tr>
<td>Nicaragua</td>
<td>1.325</td>
<td>10</td>
</tr>
<tr>
<td>Peru</td>
<td>1.6</td>
<td>10</td>
</tr>
<tr>
<td>Philippines</td>
<td>2.0</td>
<td>10</td>
</tr>
<tr>
<td>Senegal</td>
<td>1.875</td>
<td>10</td>
</tr>
<tr>
<td>Zaire</td>
<td>1.75</td>
<td>10</td>
</tr>
<tr>
<td>Zambia</td>
<td>1.425</td>
<td>10</td>
</tr>
</tbody>
</table>


*When several premiums were charged in different stages of the loan duration period, a weighted average was calculated. Also, when several loans with the same duration were observed, a weighted average was calculated.*

---

Table 2A. Transactions in the Eurodollar market: January–July, 1975*

<table>
<thead>
<tr>
<th>Borrowing country</th>
<th>Risk premium (per cent)</th>
<th>Loan duration (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algeria</td>
<td>1.375</td>
<td>7</td>
</tr>
<tr>
<td>Argentina</td>
<td>2.0</td>
<td>5.5</td>
</tr>
<tr>
<td>Bolivia</td>
<td>2.179</td>
<td>8</td>
</tr>
<tr>
<td>Bolivia</td>
<td>2.0</td>
<td>5</td>
</tr>
<tr>
<td>Brazil</td>
<td>1.75</td>
<td>5</td>
</tr>
<tr>
<td>Brazil</td>
<td>1.875</td>
<td>7</td>
</tr>
<tr>
<td>China</td>
<td>1.5</td>
<td>6</td>
</tr>
<tr>
<td>Colombia</td>
<td>1.75</td>
<td>7</td>
</tr>
<tr>
<td>Costa Rica</td>
<td>2.0</td>
<td>4.75</td>
</tr>
<tr>
<td>Ecuador</td>
<td>2.0</td>
<td>5</td>
</tr>
<tr>
<td>Ecuador</td>
<td>1.5</td>
<td>6</td>
</tr>
<tr>
<td>El Salvador</td>
<td>1.75</td>
<td>5</td>
</tr>
</tbody>
</table>
A model for analysing lenders' perceived risk

Table 2A. (cont.)

<table>
<thead>
<tr>
<th>Borrowing country</th>
<th>Risk premium (per cent)</th>
<th>Loan duration (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Guyana</td>
<td>2.5</td>
<td>5</td>
</tr>
<tr>
<td>Indonesia</td>
<td>1.725</td>
<td>5</td>
</tr>
<tr>
<td>Ivory Coast</td>
<td>1.875</td>
<td>5</td>
</tr>
<tr>
<td>Jamaica</td>
<td>1.9</td>
<td>5</td>
</tr>
<tr>
<td>Korea</td>
<td>2.0</td>
<td>5</td>
</tr>
<tr>
<td>Malaysia</td>
<td>1.625</td>
<td>5</td>
</tr>
<tr>
<td>Mexico</td>
<td>1.5</td>
<td>3</td>
</tr>
<tr>
<td>Mexico</td>
<td>1.5</td>
<td>5</td>
</tr>
<tr>
<td>Nicaragua</td>
<td>2.07</td>
<td>5</td>
</tr>
<tr>
<td>Panama</td>
<td>1.75</td>
<td>5</td>
</tr>
<tr>
<td>Panama</td>
<td>1.875</td>
<td>7</td>
</tr>
<tr>
<td>Peru</td>
<td>1.788</td>
<td>7</td>
</tr>
<tr>
<td>Philippines</td>
<td>1.755</td>
<td>6</td>
</tr>
<tr>
<td>Spain</td>
<td>1.5</td>
<td>5</td>
</tr>
<tr>
<td>Spain</td>
<td>1.5</td>
<td>7</td>
</tr>
<tr>
<td>Sudan</td>
<td>1.75</td>
<td>5</td>
</tr>
<tr>
<td>Zambia</td>
<td>2.0</td>
<td>5</td>
</tr>
</tbody>
</table>


*When several premiums were charged in different stages of the loan duration period, a weighted average was calculated. Also, when several loans with the same duration were observed, a weighted average was calculated.

REFERENCES


Optimal International Borrowing
Capital Allocation and Credit-Worthiness Control*

By Gershon Feder, Washington D. C. and Richard E. Just, Berkeley

"Official borrowers like any others, wish to borrow on the best available terms in the private market... After all, they do command the reserves of the country as well as access to domestic capital markets and the central bank. Such factors enter in their balance of payments management and, more particularly, their external debt and reserve management. They know that their external debt management strategies must be played out in the market place." (Friedman 1977a, p. 16)

The determination of appropriate international borrowing policy has attracted the attention of many economists because economic growth, particularly in developing countries, depends heavily on the inflow of foreign funds. Work on this topic dates back to Adam Smith; but more recent contributions have been made by Hamada (1966, 1969), Bardhan (1967), Van Long (1974), Manning (1972), and McCabe and Sibley (1976). A common approach in related economic analyses has involved a "small country" assumption that has been taken to imply an infinitely elastic supply of loans at a given interest rate. It has been argued recently by Hanson (1974), however, that the supply of loans may not be completely elastic even for the small country because, as debts get large, the probability of a debt crisis for the individual country increases. That is, as debts get large relative to the country (not the market), it becomes more likely that the borrower will not be able to make repayments on schedule; hence, lenders will become more reluctant to lend as the expected, discounted returns decline. As indicated by Mohammed and Saccomanni (1973), there is no doubt that this risk plays a major role in the lending practices of commercial banks. Non-profit-oriented lending institutions are also likely to be concerned with the probability that reschedulings will be required; hence, their lend-

* The views and suggestions expressed in this paper belong to the authors alone and do not necessarily reflect those of the institutions with which they are affiliated.
ing decisions may also be influenced to some degree by their borrowers' credit status (Clive and Sargan (1975)). It thus follows that even small countries do not face an infinitely elastic supply of foreign funds; rather, the terms of credit are endogenous depending on credit-worthiness. Credit-worthiness, in turn, is determined by the countries' economic performance (usually measured by a few accepted economic indicators).

Since economic-performance indicators are affected by economic policies, it is important to explore borrowing policies for optimal growth, taking into account the relationship between economic performance and terms of credit. Although empirical results suggest that several economic variables are related to debt servicing capacity, one may accept the use of only a few indicators as a useful simplification for theoretical analysis (Feder and Just (1977a, b)). Hanson has suggested the use of the debt-equity ratio for this purpose. However, in the one-sector model developed by Hanson, this approach implicitly leads to the assumption that "resources can always be costlessly allocated to obtain the necessary foreign exchange for debt servicing" [11, p. 619, footnote 10]. Hence, the only factor limiting the availability of foreign exchange is the overall productive capacity of the country which may be represented by the capital stock.

Once installed, however, capital is not a malleable factor than can be equally productive in any sector. Although some types of capital are more flexible than others (transportation equipment, power plants, etc.), it is more likely with many exports, particularly in developing countries where exports are composed mainly of nonmanufactured goods, that investments are highly specialized (mines, dams, etc.). Even when reallocation of capital is possible, the time lag involved may be too long for lenders to consider the productive capacity of other sources as relevant for the evaluation of export potential in the case of debt payment difficulty. Since the export sector is the main source of foreign exchange earnings in many countries and capital reallocation may not be possible, it thus seems that the most appropriate case for developing countries may be where the terms of credit depend specifically on the size of the export sector (as well as the debt burden). This is consistent with the findings of a recent study regarding lending behavior in the Eurodollar market where two variables related to ex-

1 Such policies have been suggested by observers of international markets; see Irvine et al. (1970), Friedman (1977 b).
Borrowing, Capital Allocation and Credit-Worthiness Control

Port performance (the ratio of debt service to export earnings and a measure of export variability) appear to be considered as relevant by lenders, while the ratio of debt to gross national product (which is a proxy for the debt-equity ratio) was found insignificant (Feder and Just 1977 a).

Another balance of payments item which is of importance in credit-worthiness evaluations is the size of foreign exchange reserves, which is usually viewed in comparison to the import bill (Feder and Just (1977 a), Goodman (1977)). That rational reserve policies are carried out with reference to credit worthiness (among other arguments) is evident from Friedman (1977 a, p. 18) who states: "Other developing countries, like Brazil, are borrowing both to finance external deficits and to rebuild reserves, profiting from the experience that a strong, visible level of reserves improves creditworthiness as well as providing an additional cushion to meet contingencies."

The purpose of the present paper is to consider a two-sector model where export and import activities are defined explicitly. The model recognizes lenders' concern regarding default on international debt through a credit-worthiness dependent supply function of foreign funds. It is assumed that capital, once allocated between the export or nonexport sector, cannot be reallocated. Accordingly the credit-worthiness assessment carried out by lenders (and which focuses on the foreign exchange attributes of the economy) is reflected in an interest rate which is positively related to the volume of anticipated debt service payment relative to the magnitude of export earnings. Similarly, the interest rate is negatively related to the size of foreign exchange reserves relative to the volume of import expenditures.

The model presents a characterization of optimal investment and borrowing decisions and discussed their policy implications in a mixed economy where both governmental and private decision making coexist.

The Model

The model below describes as small, two-sector open economy. One sector produces an export good, while the other sector produces a good which can be used for either consumption or investment. Investment

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2 This assumption has been adopted in several other optimal growth models where terms of credit and optimal borrowing are not the major interest; see works by Bose (1969), Chakravarty (1969), Dasgupta (1969) Johansen (1967), and Ryder (1969).
can be augmented with imports of capital goods. The economy has access to international capital markets, but the terms of credit it obtains depend on its credit-worthiness status. The planning authority is assumed to be interested in maximizing the discounted utility streams of consumption.

The following notation is used in the analysis. All variables should carry a time subscript, which is omitted for the purpose of brevity. A dot over a variable denotes a rate of change over time.

A. Variables

(i) Flow variables
- $F$ = A composite good which can be used for both consumption and/or investment;
- $G$ = Export good (not used domestically);
- $C$ = Consumption;
- $I_d$ = Capital formation from domestic output;
- $I$ = Total investment;
- $M$ = Quantity of imports of capital goods;
- $L$ = Gross loans;
- $U$ = Utility of consumption.

(ii) Stock variables
- $K_F, K_G$ = Stock of capital in F and G sectors, respectively;
- $B$ = Outstanding external debt;
- $R$ = International reserves.

(iii) Other variables
- $\alpha$ = The proportion of total investment which is directed to the F sector;
- $\tau$ = Interest rate on external debt.

B. Parameters
- $i$ = Rate of interest earned on foreign exchange reserves;
- $\rho$ = The planner's rate of time discount;
- $\delta$ = Rate of amortization of external debt;
- $\delta$ = Rate of capital depreciation;
- $P_g$ = International price of the export good;
- $P_m$ = International price of the capital good;
- $\bar{\tau}$ = Average interest rate as perceived by lenders.

C. Model equations

(1) $G = G(K_p) ; \ G' > 0 ; \ G'' < 0$ .
(2) $F = F(K_p) ; \ F' > 0 ; \ F'' < 0$ .
The above equations describe the production technologies of the two sectors, which utilize capital only. Labor is assumed a non-binding constraint. For simplicity, labor growth is ignored.

\[ F(K) = C + I_d. \]

Equation (3) describes the allocation of the F good output between consumption and investment.

\[ \tau = \tau \left[ G - (\theta + \tau)B, R - P_m M \right]; \tau_1 = \frac{\partial \tau}{\partial \left[ P_y G - (\theta + \tau) B \right]} < 0; \]
\[ \tau_2 = \frac{\partial \tau}{\partial (R - P_m M)} < 0; \tau(\infty, \infty) = i. \]

Equation (4) defines the endogenous interest rate on external debt as a (declining) function of two indicators of debt servicing capacity: (i) The difference between export foreign exchange earnings and the average volume of debt service payments (the latter being approximated by some perceived average rate of interest \( \tau \), which lenders expect to prevail on the average); and, (ii) The difference between international reserves held by the economy and the import bill.

These two indicators correspond to two concepts widely used in credit worthiness assessment by lenders in international capital markets: (i) the debt service pressure measure which compares the major source of foreign exchange (exports) to the most rigid expenditure item in the balance of payment (debt service); and, (ii) the import-coverage measure which compares the size of reserves to the volume of imports, thereby reflecting the extent to which imports can be financed by reserve depletion if, for some unexpected reason, foreign currency inflows (exports and new loans) are not available.

While in most cases these measures are expressed as ratios (rather than differences), it is more convenient to follow the present formulation. The economic essence of the analysis is unaffected. As both common sense and empirical evidence suggest, the interest rate will be

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3 Evidence on the prevalence of debt service and import coverage measures in credit worthiness analysis is available in many studies. See for instance Goodman (1977) who reports the results of a survey conducted among American banks engaging in international lending. Similar evidence is provided in Feder and Just (1977a) who analysed factors affecting credit terms in the Eurodollar market. See also the articles by Friedman (1977a), Brackenridge (1977), Puz (1977) and van Agtmael (1976), all of whom are senior executives in international banks.
lower for borrowers with higher exports (relative to debt service) and higher reserves (relative to imports), thus the partial derivatives $r_1$ and $r_2$ are negative. Also, for a risk-free borrower (such that both arguments of the $r$ function tend to infinity), the rate of interest will approach the going rate on international deposits ($i$).

\[ U = U(C); \quad U' > 0; \quad U'' < 0; \quad U'(0) = \infty. \]

The utility of consumption has the standard properties of utility functions, namely, positive but decreasing marginal utility, and infinite marginal utility at $C = 0$.

\[ I \equiv I_d + M. \]

Total investment is composed of domestic capital formation augmented by imports of capital goods. While the present model assumes that imports are composed of capital goods only, one can show that identical results may be derived from a model where imports are composed of consumption goods.

\[ \dot{R} \equiv (P_g G + L + iR) - [P_m M + (r + \delta) B]. \]

The difference between foreign exchange inflows (composed of exports, loans and interest earnings on reserves) and foreign exchange outflows (composed of imports and debt service payments) equals the change in international reserve holdings.

\[ K_f' = \alpha I - \delta K_f. \]
\[ K_g' = (1 - \alpha) I - \delta K_g. \]

Equations (8) and (9) present the rate of change (over time) of capital stocks in the $F$ and $G$ sectors (i.e., net investments) as the difference between the new investments allocated to each sector and the volume of capital depreciation\(^4\).

\[ \dot{B} = L - \delta B. \]

The change in indebtedness is equal to the difference between gross loans and amortization payments.

\(^4\) The proportion of investment allocated to each sector should be constrained between zero and one. It will be assumed that this condition holds at the relevant range.
The planner's objective is maximization of discounted utility of consumption. Assuming an infinite horizon the problem can be formulated as

\[ \text{Max } \int_{0}^{\infty} U(C) \, \text{dt} \]

subject to initial stock values and equations (1) - (10).

Problem (11) can be solved using the Pontryagin Maximum Principle. Denote the present value Hamiltonian by \( H \) and define the constant variables \( \lambda, \mu, \eta \) and \( \varphi \), which are dynamic shadow prices of the stock variables \( K_1, \ K_g, \ B \) and \( R \), respectively. Then, using the relation \( I = F(K_1) - C + M \) (from (3) and (5)), one can write

\[ H \cdot e^{-\vartheta t} = U(C) + \lambda \left( F(K_1) - C + M \right) - \delta K_1 \]

\[ + \mu \left( (1 - \alpha) \left( F(K_1) - C + M \right) - \delta K_g \right) + \eta \left( L - \delta B \right) \]

\[ + \varphi \left( P_\delta G(K_g) + L + \delta R \right) - \left( P_m M + (\tau + \delta) B \right) \]

It should be noted that the non-negativity requirement for \( C \) is not specified since the assumption \( U'(0) = \infty \) eliminates such a solution. The non-negativity of \( M \) and \( L \), and the requirement \( 0 < \alpha < 1 \) are assumed to be non-effective constraints at the range of the optimal path described in the present paper. It thus follows that the optimality conditions to be derived below are applicable only at a certain range of the optimal path (including the steady state). Other portions of the optimal path include corner solutions (e.g., zero investment in one sector or the other, zero borrowing, etc.) and seem to us to be less interesting from a practical point of view. Similarly, existence and uniqueness of the solution depend on additional constraints which are not spelled out.

The optimality conditions require (in addition to appropriate transversality conditions), at the range where all controls are positive:

\[ \frac{\partial H}{\partial C} = 0 \quad \Rightarrow \quad U' - \alpha \lambda - (1 - \alpha) \mu = 0 \]

\[ \frac{\partial H}{\partial M} = 0 \quad \Rightarrow \quad \alpha \lambda + (1 - \alpha) \mu - \varphi P_m [1 - \tau_2 B] = 0 \]

\[ \frac{\partial H}{\partial \alpha} = 0 \quad \Rightarrow \quad [F(K_1) - C + M] \cdot (\lambda - \mu) = 0 \]

\[ \frac{\partial H}{\partial L} = 0 \quad \Rightarrow \eta + \varphi = 0 \]
Analysis of the Results

The economic interpretation of the optimality conditions is fairly straightforward:

Equation (13) implies that the marginal benefit of consumption (i.e., marginal utility) equals its opportunity cost \([\alpha \lambda + (1 - \alpha) \mu]\), which is the marginal value of investment foregone. The value of marginal investment is a weighted average of the value of capital in both production sectors. As will become apparent, the value of a unit of capital in both sectors is equal, thus the implication is that the marginal utility of consumption equals the value of a unit of capital which is foregone.

Equation (14) states that the value of a unit of investment \([\alpha \lambda + (1 - \alpha) \mu]\) (afforded by one additional unit of imports of capital goods) equals its opportunity cost. The latter is the amount of foreign exchange (withdrawn from reserve stocks), adjusted for the deterioration in credit terms caused by the decline in import coverage by reserves. We note that if imports had no effect on the term of credit (i.e., \(r_2 = 0\)) then the value of investment equals the shadow price of foreign exchange needed to implement that investment.

Equation (15) implies that the value of capital in each of the two production sectors (that is, the export sector and the investment-consumption sector) is equal, (i.e., \(\lambda = \mu\)) even though capital is not transferable between the sectors once investments are implemented (except in the case \(F(K) - C + M = 0\), which implies no investment in the economy). Given \(\lambda = \mu\), it follows that the value of capital is higher than the value of the direct foreign exchange cost of a unit of capital imports. This is explained (as indicated above) by the additional cost, in terms of reduced credit worthiness, of imports.

Equation (16) implies that the value of a unit of reserves (or the shadow cost of foreign exchange) equals its opportunity cost (the
reduction of external debt). The value of a unit debt ($\eta$) is obviously negative, as the value of reserves ($q'$) is positive.

Using the result $\lambda = \mu = \varphi P_m (1 - r_2 B)$ (which also implies $\dot{\lambda} = \dot{\mu}$), equations (17) and (18) can be combined to yield

\[
\frac{P_m F'}{P_g G'} = \left(\frac{1 - r_1 B}{1 - r_2 B}\right).
\]

Equation (21) implies that in general the value of marginal product of capital in the two production sectors (evaluated at international prices) will not be identical. Obviously, in the case where no credit worthiness considerations are taken into account (i.e., $r_1 = r_2 = 0$), the marginal productivity of capital would be maintained equal in all sectors. Indeed, in an economy where most investments are made by private (and competitive) investors, one would expect them to operate such that $P_m F' = P_g G'$, since each one of them is too small to consider the national credit-worthiness implications of imports and exports. The planner (or the government), however, cannot ignore the impact of such variables as represented by the right hand side of equation (21). Thus, intervention in the sectoral allocation of investment is called for by export promotion and/or import substitution. Export promotion is reflected in the term $-r_1 B$ which measures (in units of output) the amount of subsidy that should be given to producers of the export good. Similarly, $-r_2 B$ measures (in physical units) the subsidy due to producers of the $F$ good (which can substitute imports of capital goods). Private investors will then equate (at the margin) the actual rates of return received by them, namely

\[
P_g G' \cdot [1 - r_1 B] = P_m F' \cdot [1 - r_2 B],
\]

which will ensure that the economy maintains the optimal condition given by equation (21). At the steady state, however, the optimal condition for the $F$ sector requires (from equation (17) with $\dot{\lambda} = 0$),

\[
F' = \delta + Q.
\]

Whether the export sector is (at the margin) more profitable than the import substituting sector depends on whether $r_1/r_2 \geq 1$. The size of $r_1/r_2$ is affected by the differential weight attached to the different credit-worthiness indicators and is thus an empirical question.

It should be noted that in practice, many governments find it more convenient to control the volume of imports (by tariffs or quotas), so as
to avoid foreign exchange crises which will deteriorate their credit status. In terms of the present model, the optimal tariff on imports is $-\tau_2 B$. Imposing such an import tax will eliminate the need to subsidize the $F$ sector.

Using the equality $\eta = -\varphi$, which implies $\dot{\varphi}/\varphi = \dot{\eta}/\eta$, equations (19) and (20) can be combined to yield

\begin{equation}
   r - r_1 (\vartheta + \bar{\tau}) B = i - \tau_2 B .
\end{equation}

The left hand side of equation (24) is the marginal cost of borrowing to the economy, which is necessarily higher than the rate of interest $r$, since it includes the extra cost due to the deterioration in terms of credit brought about by higher debt (the latter effect is represented by the term $-\tau_1 \cdot (\vartheta + \bar{\tau}) \cdot B$). The right hand side represents the marginal benefit from reserve holdings, which is composed of the direct rate of interest plus the contribution to credit worthiness gained by additional reserves (given by $-\tau_2 B$). Equation (24) thus describes the optimality relation for reserve and debt stocks. Obviously, in the present model, if reserves had no impact on credit worthiness ($\tau_2 = 0$), then a zero reserves level should be maintained. This can be seen by observing that with $\tau_2 = 0$, it is impossible for equation (24) to hold (since $r - r_1 (\vartheta + \bar{\tau}) B > i$ for all values of the arguments of $r$, as indicated in (4)). To approach the equality, $R$ will be depleted (and $B$ reduced by an equal amount), which verifies the assertion made above.

In an economy where part of the external borrowing is done by private corporations, the model points out the necessity of imposing a tax on foreign borrowing. The reason for such a tax is that while the cost of borrowing for each individual borrower is $r$ (the rate of interest) the cost to the economy is the marginal borrowing cost which includes the terms of credit effect. The presence of credit worthiness considerations thus implies an externality, which may be corrected by an appropriate tax. Obviously, direct measures can also be applied. For instance, recently, the government of Turkey banned foreign loans which are obtained at a rate of interest higher than a certain upper limit (Clarke, 1977). Other governments imposed administrative restrictions which amount to an implicit tax on borrowing from foreign sources.

Comparisons between the marginal rate of return and the external interest are possible, in the present model, only at the steady state. Thus, setting $\dot{\lambda} = \dot{\eta} = 0$ and using the results $\lambda = \mu$, $-\eta = \varphi$, one obtains from (17) and (19):
Equation (25) implies that at the steady-state, the marginal productivity of capital in the import-substituting sector (net of capital depreciation allowance) should be maintained at a level higher than the rate of interest of external debt. The difference is equal to the credit-worthiness effect of marginal borrowing. When credit-worthiness considerations are negligible ($r_1 \approx 0$), the standard result is obtained, namely, the net marginal productivity of capital equals the interest rate.

The relation between the marginal rate of return of capital in the export sector and the rate of interest is less straightforward. Setting $\dot{\mu} = \dot{\eta} = 0$, and using the relation $- \eta = \varphi = \mu/[P_m \cdot (1 - r_2 \cdot B)]$, one obtains from (18) and (19), at the steady state,

$$
\frac{P_g G'}{P_m} \cdot \frac{(1 - r_1 B)}{(1 - r_2 B)} - \delta = r - r_1 (\bar{r} + \bar{\varphi}) \cdot B.
$$

Since the magnitude of the various terms on the left hand side is not known a-priori, one cannot conclude as to whether the rate of return to capital should optimally be higher or lower than the direct rate of interest. We note, however, that the left hand side describes the full impact of the marginal investment in the export sector: the direct contribution to export plus the improvement in credit terms induced by higher export $[P_g G'(1 - r_1 B)]$ discounted by the total cost of a unit of capital imports (that is, both direct and indirect cost). Equation (26) thus states that the full marginal contribution of investment in the export sector should equal the marginal cost of borrowing.

Setting $\dot{\eta} = 0$ also implies (by equation (19)) that at the steady state the rate of time preference of the economy ($\phi$) exceeds the interest rate on external debt, as it needs to equal the marginal cost of borrowing.

From equation (17) with $\dot{\lambda} = 0$, one obtains the familiar “golden rule” of neo-classical models, asserting that at the steady state capital accumulation is maintained at a level such that the net marginal productivity equals the rate of time preference (if labor growth is nil). This rule applies, in the present model, only to the import substituting sector. For the export sector the optimal steady state accumulation law is more complicated and depends on credit-worthiness effects.
Conclusion

By way of summary, the model implies the following set of policies for an economy with private borrowing, investment and production:

(i) Export promotion through a subsidy (of magnitude $- P_g r_1 B$) to the export sector;

(ii) Tax on capital goods imports (of magnitude $- P_m r_2 B$), which amounts to a policy of import substitution;

(iii) Tax on foreign borrowing by private corporations (of magnitude $r_1 (\theta + \eta) B$).

These recommendations are, of course, of a partial nature, due to the simplistic structure of the model. But the paper makes the general argument that development policies should take into account the relationship between economic policies and the supply of foreign funds, and it demonstrates that such considerations require central government intervention in the operation of the market. This conclusion is most relevant for developing economies who depend on commercial international capital markets for a substantial portion of their foreign exchange inflow.

References

Zusammenfassung

Optimale internationale Schuldenaufnahme
Kapitalallokation und Kreditwürdigkeitskontrolle

Summary

Optimal International Borrowing, Capital Allocation and Credit-Worthiness Control

The paper develops a two-sector model of a growing economy, incorporating the relation between debt servicing capacity and the terms of credit facing the country. These terms are affected by macro-economic variables such as the volume of imports and exports, the size of foreign exchange reserves and the magnitude of debt service payments due on outstanding debt. The fact that importers, exporters and producers may be too small to take account of the impact their actions have on the economy’s terms of credit creates discrepancies between private and public optimal solutions. The model suggests that optimal growth may require export promotion while simultaneously taxing both private borrowing of foreign funds and importation of capital goods.

Résumé

Endettement international optimal, allocation de capital et contrôle de la solvabilité

La présente étude développe un modèle bisectoriel d’économie en expansion, incluant la relation entre la capacité d’endettement et les conditions de crédit d’un pays. Ces conditions sont influencées par des variables macro-économiques telles que le volume des importations et des exportations, le stock des réserves de devises, le montant des paiements du service de la dette correspondant à l’endettement en cours. Le fait que les importateurs, les exportateurs et les producteurs n’ont fréquemment pas la taille requise pour influer par leur manière d’agir sur les conditions du crédit de l’économie provoque des disparités entre les endettements optimaux privé et public. Le modèle conclut que la croissance optimale requiert la promotion des exportations et parallèlement la taxation des emprunts privés en devises étrangères et de l’importation de biens d’investissement.
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