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Multidimensional and Specific Inequalities

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Abstract

Despite the multitude of measures of multidimensional inequality, none is regularly used in policymaking. This paper proposes multidimensional inequality measures that are easily implementable and transparent and overcome many deficiencies of existing measures. The measures follow a traditional two-stage format, which aggregates dimensions first and then applies a unidimensional measure like the Gini coefficient to the distribution of aggregates. A novel characterization result identifies the precise form of aggregation needed to obtain axiomatically sound measures. The paper derives an additive decomposition formula—breaking down multidimensional inequality into terms reflecting the average specific inequalities (within dimensions) and the joint distribution (across dimensions)—for any measure created using a standard unidimensional measure or the Lorenz curve. The paper also provides an approach to calibrating the measure for use with data over time, replacing the usual ad hoc normalization of variables with one that accounts for a policymaker's normative weights. The technology is illustrated first using synthetic data to understand how the measure varies as the components are changed and then using data from Azerbaijan.

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Multidimensional and Specific Inequalities*

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I. Introduction

While global income inequality has declined since the 1990s, income inequality levels within countries have been rising for a wide range of developed and developing countries, capturing the attention of social activists and policymakers alike. More than half of all countries and close to 90 percent of advanced economies have seen an increase in income inequality since 2000, with the income Gini increasing by more than two points in some instances (IMF 2023). Yet inequality is present not only in the space of incomes; it inhabits other key dimensions such as health, education, and social services, whose dimension-specific inequalities may reinforce or dampen the impact of income inequality. Empirical data on income inequality offers only a partial view of what Amartya Sen has termed "economic inequality" and can limit the scope and accuracy of a country's policy responses (Sen 1997, 1999).

Dashboards and weighted averages of dimension-specific inequalities can help paint a broader picture of economic inequality within a country, and its evolution through time. However, they completely ignore the joint distribution of dimensional variables, which conveys important information on how people in the country are experiencing inequality. For example, it could be the case that most people with lower levels of one variable have lower levels of the others, resulting in a rigid hierarchy of achievement vectors and, arguably, greater economic inequality. Alternatively, people could exhibit mixed levels of achievements, dampening positive association and its impact on inequality. Measurement tools sensitive to the joint distribution can distinguish between these situations, and better gauge the extent of economic inequality.[1]

Following the pioneering work of Kolm (1977), Atkinson and Bourguignon (1982), and Maasoumi (1986), there have been significant advances in the range of tools available for measuring inequality when there are multiple dimensions, including new classes of measures, axioms for discerning among measures, and dominance methods for ensuring comparisons are robust.[2] In their survey on multidimensional poverty and inequality, Aaberge and Brandolini (2015, p. 201) note a strong demand for multidimensional analyses by policymakers and other stakeholders, and indeed, empirical and policy applications of multidimensional poverty indices

[1] For Aaberge and Brandolini (2015, p. 146) this sensitivity is "the single feature that distinguishes multidimensional from unidimensional analysis." Note that it also places additional requirements on the data that can be used.
[2] See the recent surveys of Aalberg and Brandolini (2015), Andreoli and Zoli (2020), Glassman (2019), and Seth and Santos (2018).

(MPIs) are numerous.[3] However, for multidimensional *inequality* measures, the policy impact has been more muted.

Why is this? One possibility might be the complexity of existing measures.[4] To be effective in policy analysis, a measure needs to be easily understood and communicated. But this desirable characteristic is unlikely to be present unless it has been prioritized and intentionally built into the measure along with the traditional axiomatic requirements.[5] In addition, the process of bringing a particular multidimensional measure to data can be daunting, requiring many consequential choices not addressed in most theoretical presentations (Alkire and Foster 2010). On what basis should a given cardinalization of a variable be selected? How can the variables be made comparable to one another? Where should the relative importance of variables be reflected? The structure of a measure might facilitate or hinder the answers to such questions, impacting its dependability and ease of use in policy analyses.

The aim of this paper is to identify axiomatically-sound multidimensional inequality measures having attributes well-suited for policy.[6] Our focus is the two-stage approach of Maasoumi (1986), which yields intuitive measures frequently used in empirical analyses.[7] In this approach, a multidimensional inequality measure is constructed using two components: an *aggregation function* converting each person's dimensional achievements into an aggregate indicator; and an *inequality measure* evaluating the resulting vector of indicators. His original presentation used general means and generalized entropy measures; ours considers a general set of aggregation functions based on Bosmans et al. (2015) and Lorenz-consistent inequality measures.[8]

Two-stage measures inherit several of the standard axioms for multidimensional inequality measures from the properties of their components. However, Dardanoni (1995) has shown that at

[3] Google Scholar and the site www.mppn.org list hundreds of empirical studies using MPIs along with many policy applications. The relative infrequency for multidimensional inequality indices is noted in Hong (2009), Seth and Santos (2018) and IPSS (2018).
[4] For example, Lugo (2007) notes how the parameters of a measure can obscure its meaning, hampering applications.
[5] See Foster (2024) who discusses an approach to "intentional measurement".
[6] As we shall see below, the required properties include anonymity, scale invariance and replication invariance, as well as the two generalizations of the transfer principle to multidimensional measures.
[7] According to Bosmans, Decancq, and Ooghe (2015 p. 95) the two-stage approach "dominates the empirical literature". Recent applications include Nilson (2010), Justino (2012), Rhode and Guest (2013, 2018), Bartels and Stockhausen (2016).
[8] Aggregation functions are assumed to satisfy continuity, concavity, and linear homogeneity; Lorenz-consistent measures follow the Lorenz criterion when it applies.

least one property is not assured: Kolm's (1977) weak majorization axiom, which requires the level of multidimensional inequality not to rise when each dimensional distribution is "smoothed" using the same bistochastic matrix. This is a fundamental axiom that generalizes the Pigou-Dalton transfer principle to the multidimensional context. Consequently, additional restrictions on the two components may be needed to ensure that the resulting measures are axiomatically sound.

Our first results characterize the subset of measures satisfying the standard multidimensional axioms, including weak uniform majorization. We show that, while any Lorenz-consistent inequality measure can be used at the second stage, the only form of aggregation that can be used in the first is linear. Given this specification, we show that the two-stage measures satisfy all the basic axioms, including, perhaps surprisingly, the unfair rearrangement axiom, which ensures that a multidimensional inequality measure is appropriately sensitive to positive association among the variables. The Lorenz curve can also be applied in the second stage to obtain a graphical depiction of multidimensional inequality and a dominance criterion that indicates when all two-stage measures with the same aggregation would agree on a comparison.

The next series of results explores the link between multidimensional inequality and the (dimension-)specific inequalities. Following Shorrocks (1978), we consider Lorenz consistent measures satisfying a basic convexity assumption.[9] We show that multidimensional inequality can be expressed as a weighted average of specific inequalities minus a non-negative term reflecting the relevant aspects of the joint distribution across dimensions. In the case of the Gini coefficient (or the Lorenz curve) this final term is particularly intuitive: it is the extent to which multidimensional inequality would rise if achievements were completely aligned.

To implement the measure, we provide a calibration approach based on data in an initial period and normative policy weights. An average Lorenz curve is constructed by weighting and summing up the specific Lorenz curves for the country in the initial period. Then, working back through the Lorenz formula, coefficients for the linear aggregation function are extracted to reflect the normative weights. In essence, dimensions are rendered comparable using the measuring rod of Lorenz (or Gini) inequality.[10] Once the multidimensional inequality measure

[9] The property of constant-sum convexity is satisfied by virtually all traditional measures.
[10] This is analogous to the role of deprivations in multidimensional poverty (Alkire et al 2015, p. 50).

has been calibrated using the initial period's data, it can then be used to gauge the country's multidimensional inequality through time and, with additional assumptions, through space.

We illustrate the methodology using simulated data to allow specific inequalities, distributional means, and correlation levels to vary freely. Examples show the impact of each factor on multidimensional inequality for measures based on the Lorenz curve, the Gini coefficient, and other Lorenz-consistent measures. A second illustration using data from Azerbaijan examines the evolution of multidimensional inequality during a period of rapid income growth.

Previous studies have considered linear aggregation as one among several options, but to our knowledge this is the first paper that selects this structure based on the axiomatic properties of its associated measures. Likewise, a number of authors have linked multidimensional inequality to specific inequalities or to positive association, but none has the elegant simplicity of our decomposition, which highlights the fundamental role of Shorrocks mobility in representing a positive association in multidimensional inequality. In addition, the approach is unique in its use of normative weights and Lorenz curves to calibrate the measure in a base year, and then to judge subsequent changes accordingly. Finally, the paper is unusual in its focus on identifying multidimensional inequality measures that are especially useful in conducting policy.

Section II introduces the definitions and notation used in the paper. Section III presents different approaches to multidimensional inequality measurement and then establishes our main characterization result, identifying the subset of measures from Maasoumi's (1986) two-stage approach that are axiomatically sound. Section IV provides expressions linking members of this class of multidimensional inequality measures to specific inequalities and Shorrocks mobility. Our calibration method is described in Section V, selecting initial parameters of the measure based on normative weights and Lorenz curves. Section VI illustrates our methods, while the final section concludes.

II. Notation, Axioms, and Other Fundamentals

The data for measuring inequality are given in an array x with n rows and d columns. The i^{th} row x_i lists data for person $i = 1, \dots, n$; the j^{th} column $x_{\cdot j}$ lists data for dimension $j = 1, \dots, d$; and each entry $x_{ij} \geq 0$ is person i's achievement in dimension j. The population size n can vary, but d is fixed in any given application; hence, the set X of possible arrays is a subset of

$\cup_{n=1}^{\infty} \mathrm{R}_{+}^{nd}$. Two alternatives for X will be considered: $X = X_1$ containing arrays that are strictly positive, and $X = X_2$ containing nonnegative arrays having at least one positive quantity in each column.[11] A *multidimensional inequality measure* is a mapping $M: X \to R$ associating a real number to each array, interpreted as its level of multidimensional inequality.

When developing a measurement tool like M, it is important to be clear about its intended purpose, its desired characteristics, and the axioms it should satisfy (Foster 2024). The purpose of the measure we seek is to monitor multidimensional inequality in a country over time. A useful list of desired characteristics (or desiderata) can be found in Szekely (2006); of special relevance to the present paper is one that calls for a measure to be understandable and easy to describe.[12] The list of axioms to be satisfied by a multidimensional inequality measure will be given below.

We make use of unidimensional inequality measures in producing and understanding multidimensional measures. Let $V^{\vdots}$ denote the set of column vectors of arbitrary length associated with X. A *unidimensional inequality measure* is a mapping $I: V^{\vdots} \to R$ associating a real number $I(v)$ to each vector v in $V^{\vdots}$ interpreted as its inequality level. The *Lorenz curve* $L_v: [0,1] \to [0,1]$ associated with v depicts its level of equality (and inequality), where $L_v(p)$ is the share of the income received by the lowest p share of the population. Distributions $v, v' \epsilon V^{\vdots}$ have the same level of inequality by the Lorenz criterion if $L_{v'}(p) = L_v(p)$ for all p; distribution v' has no less inequality than v if $L_{v'}(p) \leq L_v(p)$ for all p, in which case we say that v *weakly Lorenz dominates* v'; and v' has greater inequality than v if $L_{v'}(p) \leq L_v(p)$ for all p, and $L_{v'} \neq L_v$, in which case we say that v *(strictly) Lorenz dominates* v'. A measure I is said to be *Lorenz consistent* if it renders the same judgment as the Lorenz criterion when the criterion applies; equivalently, I satisfies the axioms of anonymity, scale invariance, replication invariance, and the (Pigou-Dalton) transfer principle (Foster 1985). A measure I is said to be *constant-sum convex* if it is convex over slices of $V^{\vdots}$ having the same population size and the same total.[13]

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[11] In symbols, $X_1 = \cup_{n=1}^{\infty} \mathrm{R}_{++}^{nd}$ while $X_2 = \cup_{n=1}^{\infty} (\mathrm{R}_{+}^{n} \backslash 0)^{d}$, where $\mathrm{R}_{+}^{n} \backslash 0$ denotes the nonnegative orthant in R^n excluding its origin.

[12] Other desiderata include: conforming to a common-sense notion of what is being measured; fitting the stated purpose; being technically solid; being operationally viable; and being easily replicable. See also Alkire et al (2015) and Foster (2024). Such "proto-axioms" are less precise than formal axioms but help ensure a measure is fit for purpose.

[13] See Kolm (1976, p. 93). Like decomposability, this property constrains the cardinal values of a measure.

Applying I to column $x_{.j}$ in x yields the inequality level $I(x_{.j})$ for dimension $j = 1, \ldots, d$, which will be termed its *specific* inequality level; analogously $L_{x_{.j}}$ (or L_j when x is understood) graphically depicts its specific inequality level.

Let v be an element of $V^{\vdots}$. We denote the *mean* of v by $\mu(v) = \sum_{i=1}^{n} \frac{v_i}{n}$ or, more succinctly, by μ_j if $v = x_{.j}$. The *ordered* vector $\hat{v}$ of v is the permutation of v for which $\hat{v}_1 \leq \hat{v}_2 \leq \cdots \leq \hat{v}_n$. Given array $x \in X$, the *completely aligned array* $\bar{\bar{x}}$ of x is the array whose columns are $\bar{\bar{x}}_{.j} = \hat{x}_{.j}$; it takes the same dimensional achievements and reorders them so that person 1 has the lowest entry for each dimension, person 2 has the next lowest, and so forth. An array is said to be *aligned* if it has the same rows as its completely aligned version, but potentially in a different order.

Some measurement approaches use a function to tally up the dimensional achievements of a given person. Let $V^{\cdots}$ denote the set of row vectors of length d associated with X. An *aggregation function* is a mapping $h: V^{\cdots} \to R$ associating a real number to each vector in $V^{\cdots}$; applying h to x_i in x yields person i's *aggregate level* $h(x_i)$. Given vectors $v, v' \in V^{\cdots}$, we define the *vector dominance* relations as follows: $v \gg v'$ iff $v_j > v_j'$ for all j; $v \geq v'$ iff $v_j \geq v_j'$ for all j; and $v > v'$ iff $v \geq v'$ and not $v' \geq v$. The rows of an aligned array can be ranked by vector dominance; and in the completely aligned array the later rows vector dominate earlier rows, so that $\bar{\bar{x}}_{i'} \geq \bar{\bar{x}}_i$ for $i' > i$.

We consider two categories of axioms for multidimensional inequality measures – invariance axioms and dominance axioms. *Invariance* axioms specify the transformations of an array that leave the measure unchanged. They include *anonymity*, *scale invariance*, and *replication invariance* and are entirely analogous to the unidimensional versions. *Dominance* axioms specify transformations that cause multidimensional inequality to move in a certain direction. They include the two multidimensional generalizations of the Pigou-Dalton transfer principle associated with Kolm (1977) and Atkinson and Bourguignon (1982), respectively.

Intuitively, Kolm's (1977) weak uniform majorization axiom requires multidimensional inequality not to increase when each dimension is "smoothed" in the same way. More precisely, we say that x' is obtained from x by a *uniform smoothing* if $x' = Bx$ for some bistochastic matrix B.[14] Under a uniform smoothing, person i's vector in x' is a weighted average of all persons' initial vectors in x, where the weights are the elements of the i^{th} row of B. This creates a new array whose columns weakly Lorenz dominate the respective columns of the original array. The first dominance axiom for M is given by the following.

Weak Uniform Majorization Axiom. If x' is obtained from x by a uniform smoothing, then $M(x') \leq M(x)$.

This axiom specifies that multidimensional inequality should not increase as a result of uniform smoothing.

When might we expect multidimensional inequality to strictly fall as a result of a uniform smoothing? When will it stay the same? We say that x' is obtained from x by a *permutation* if $x' = \Pi x$ for some permutation matrix Π.[15] Note that a given uniform smoothing could also be a permutation if, say, B were itself a permutation matrix or if the rows averaged by B happened to be identical. Since anonymous multidimensional inequality measures are unchanged by a permutation (and anonymity is typically assumed), the strict version of the axiom accounts for this possibility.

Uniform Majorization Axiom. If x' is obtained from x by a uniform smoothing, then $M(x') \leq M(x)$; if, in addition, x' is not obtained from x by a permutation, then $M(x') < M(x)$.

Many multidimensional measures have *two stages*: the first employs an aggregation function $h(x_i) = s_i$ and a second applies some symmetric function to the aggregate distribution $s = (s_1, \ldots, s_n)$.[16] It could be argued that when dealing with this form of measure, the strict form of the uniform majorization axiom needs to be modified further. To be sure, if x' is a permutation of x, then s' will be a permutation of s. However, the converse need not be true. And if it were not, then the axiom would require $M(x') < M(x)$ (since x' is not a permutation of x) at the same time that anonymity would be requiring $M(x') = M(x)$ (since s' is a permutation of s). The following modification accounts for this issue.

[14] A bistochastic matrix is a square nonnegative matrix whose rows and columns sum to 1.
[15] A permutation matrix is a square matrix containing 0's and 1's whose rows and columns sum to 1.
[16] Examples of papers that aggregate rows first include Tsui (1995, 1999), Bourguignon (1999), Diez et al. (2007), Decanq and Lugo (2012), and Seth (2013).

Limited Uniform Majorization Axiom. If x' is obtained from x by a uniform smoothing, then $M(x') \leq M(x)$; if, in addition, s' is not obtained from s by a permutation, then $M(x') < M(x)$.

This revised axiom only requires strict inequality to hold when the aggregate vectors are not permutations of one another. It should be noted that this axiom is tailor-made for two-stage multidimensional inequality measures, and it is a joint restriction on M and h.[17]

The second type of dominance axiom, based on Atkinson and Bourguignon (1982), takes into account the association among variables.[18] We say that x' is obtained from x by an *unfair rearrangement* if $x' = \bar{\bar{x}}$, where $\bar{\bar{x}}$ is the completely aligned version of x. An unfair rearrangement reassigns achievement levels to people so that person 1 has the lowest level in each dimension, person 2 has the next lowest levels, and so forth, thereby maximizing positive association among variables. The following is a second multidimensional generalization of the transfer axiom.

Weak Unfair Rearrangement Axiom. If x' is obtained from x by an unfair rearrangement, then $M(x') \geq M(x)$.

According to this axiom, reallocating achievements so as to maximize positive association should not decrease multidimensional inequality.

Note that as before, this transformation may yield an array that is a permutation of the original array, as would happen if x had the same rows as $\bar{\bar{x}}$, but in a different order across people. The strict version of this axiom accounts for this possibility.

Unfair Rearrangement Axiom. If x' is obtained from x by an unfair rearrangement, then $M(x') \geq M(x)$; if, in addition, x' is not obtained from x by a permutation, then $M(x') > M(x)$.

This axiom goes beyond the weaker version by requiring multidimensional inequality to strictly increase when the unfair rearrangement is not a permutation of the original array.

III. Multidimensional Inequality Measures

[17] Thus, its use is limited to the class of two-stage measures.

[18] Our presentation follows Dardanoni (1995). For other versions of the axiom see, for example, Tsui (1995).

We now present several intuitive approaches to measuring multidimensional inequality and the properties they satisfy. A *dashboard* $D(x) = \big(I(x_{\cdot 1}), \dots, I(x_{\cdot d})\big)$ is a vector of specific inequalities, which can be interpreted as a multidimensional inequality measure (or rather a quasiordering on X) when used with vector dominance.[19] Multidimensional inequality is then judged to be higher when one specific inequality level is higher, and the rest are no lower. So long as I is Lorenz-consistent, D satisfies all but one of the general axioms required of a multidimensional inequality measure.[20] The unfair rearrangement axiom fails, since D ignores information on positive association and considers $\bar{\bar{x}}$ and x to be identical. Of course, the practical utility of D is also hampered by its inability to make comparisons when one specific inequality rises and another falls.[21]

A simple way of moving from a dashboard to a multidimensional inequality measure is to take a weighted average of specific inequalities using positive weights $\omega_1, \dots, \omega_d$ that sum to 1, resulting in an *average specific inequality* measure

$$A(x) = \omega_1 I(x_{\cdot 1}) + \cdots + \omega_d I(x_{\cdot d}) \tag{1}$$

Gajdos and Weymark (2005, p. 489), for example, use the Gini coefficient and weights $\omega_j = \mu_j / \sum_k \mu_k$ to obtain a measure of this form, which they contrast to Koshevoy and Mosler (1997) who consider fixed weights. Whether weights depend on means or are fixed, A satisfies the same list of axioms as D when I is Lorenz-consistent.[22] Unlike a dashboard, an average specific inequality measure can make comparisons between any two arrays, but it also ignores the association between dimensions. In particular, it views $\bar{\bar{x}}$ and x as identical and violates the unfair rearrangement axiom.

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[19] Dashboards can also be populated by distinct inequality measures or applied to unrelated sample populations. The above case fits best in the present context. A quasiordering is a reflexive and transitive relation, that is not necessarily complete (Sen 1997 Ch 3).
[20] In particular, dashboard inequality levels are unchanged by permutations, by a scalar multiple, by a population replication, and by an unfair rearrangement; they do not rise, and can fall, as a result of a uniform smoothing. Consequently, the quasiordering generated by D satisfies the three invariance axioms, the uniform majorization axiom, and the weak unfair rearrangement axiom.
[21] Even when D *can* compare two arrays, the comparison might go against judgments that take into account information on dimensional means (and dependence); the conclusions rendered by dashboards are partial and not unambiguous.
[22] In particular, A satisfies the three invariance axioms, the weak unfair rearrangement axiom, and the uniform dominance axiom, where the latter property holds since a uniform smoothing leaves weights unchanged.

Maasoumi (1986) constructs a multidimensional inequality measure that reverses the order of aggregation by first combining a person's dimensional achievements into a single aggregate indicator $s_i = h(x_i)$ and then applying a unidimensional inequality measure I to the aggregate distribution $s = (s_1, \dots, s_n)' \epsilon\ V^{\vdots}$. The resulting *two-stage* measure $M(x) = I(s)$ is intuitive in structure, with components I and h that can be readily understood and applied.[23] However, questions about the axiomatic suitability of the approach have been raised. Dardanoni (1995) shows how a two-stage measure can violate the weak uniform majorization axiom, which leads him to critique the axiom; Weymark (2006) reinterprets this finding as a critique of Maasoumi's two-stage approach.

Bosmans et al. (2015) provide a novel justification of two-stage measures in the context of normative inequality measurement, which views multidimensional inequality as the welfare loss from falling short of an optimal allocation.[24] They divide each normative multidimensional inequality measure into two distinct terms: one that evaluates inefficiency and another that evaluates inequity and show that the latter term is, in fact, a two-stage inequality measure. The authors conclude: "If one would insist that inequality measures should be concerned with inequity alone, and not with inefficiency, then we arrive at the striking conclusion that the normative approach itself pushes two-stage measures to the forefront." This is a remarkable observation, which sheds light on the structure of normative multidimensional inequality measures as well as the suitability of the two-stage class. Note, though, that it justifies the two-stage measures not as independent multidimensional inequality measures but as useful "partial" indices that focus on one aspect of multidimensional inequality.[25]

The broader suitability of the two-stage measures depends on the axioms they satisfy, which in turn depends on the range of components h and I being considered. For the first component, we consider all aggregation functions $h: V^{\cdots} \to R$ that are continuous, concave, linear homogenous, and strictly increasing (as in Bosmans et al., 2015), and denote the resulting set by $\mathcal{H}$. For the second, we consider all Lorenz-consistent unidimensional inequality measures $I: V^{\vdots} \to R$, and

[23] While many, including Maasoumi (1986) and Bosemans et al (2015), interpret the aggregation function as utility, here we are "making no use of information on individual relative valuations" of dimensional variables, and instead are treating the function as "a subject for social decision" (Atkinson and Bourguignon 1982 p. 190).
[24] See also Kolm (1977), Tsui (1995), and Weymark (2006) for traditional derivations of a (relative) normative multidimensional inequality measure from a welfare function.
[25] On partial indices, see Foster and Sen (1997, p. 168-9).

denote the set by $\mathcal{I}$. The object of study is $\mathcal{M}$, the set of two-stage measures $M: X \to R$ with components $h \in \mathcal{H}$ and $I \in \mathcal{I}$. Which axioms are satisfied by these measures? Can we identify a subclass of $\mathcal{M}$ that is both intuitive and axiomatically sound?

The properties defining $\mathcal{H}$ and $\mathcal{I}$ ensure that every measure in $\mathcal{M}$ satisfies the axioms of anonymity, scale invariance, and replication invariance.[26] Our first result takes up the weak uniform majorization axiom.

Theorem 1. Let M be a two-stage measure with components $h \in \mathcal{H}$ and $I \in \mathcal{I}$. If M satisfies weak uniform majorization, then there exists $c = (c_1, \dots, c_d) \gg 0$ such that $h(v) = c_1 v_1 + \cdots + c_d v_d$ for all $v \in V^{\cdots}$.

Proof. See the Appendix.

The proof draws on Dardanoni (1995) and begins by showing that any convex combination of allocations in $V^{\cdots}$ having the same value under h also has the same value under h. Applying this to a certain set of allocations yields a simplex over which h is linear, while the remaining argument expands the characterization to all of $V^{\cdots}$. Theorem 1 identifies the two-stage measures that are consistent with weak uniform majorization; the remaining measures in $\mathcal{M}$ violate this basic axiom and hence are not axiomatically sound.[27]

Let $\mathcal{L}$ be the subclass of $\mathcal{M}$ whose aggregation functions are linear with $c \gg 0$. The next result describes the axioms satisfied by the measures in $\mathcal{L}$.

Theorem 2. Any two-stage measure $M \in \mathcal{L}$ satisfies the anonymity, scale invariance, replication invariance, limited uniform majorization, and unfair rearrangement axioms.

Proof. See the Appendix.

The proof shows how each invariance property for M follows immediately from the analogous property for I. For the limited uniform majorization axiom, when a bistochastic matrix is applied to array x, the new aggregate vector can be found by applying the same bistochastic matrix to the

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[26] See Theorem 2, below.
[27] This includes two-stage measures using other CES-type aggregation functions suggested by Maasoumi (1986) and used in empirical applications.

aggregate vector s of x, due to the linearity of h. Consequently, both the weak and strict parts of the axiom follow directly from the transfer axiom and anonymity of I. As for the unfair rearrangement axiom, the linear structure of h might lead one to think that the resulting measure would not be sensitive to the joint distribution and hence that $M(\bar{\bar{x}}) = M(x)$. Yet the proof shows that $M(\bar{\bar{x}}) > M(x)$ follows immediately from the Lorenz consistency of I whenever $\bar{\bar{x}}$ is not a permutation of x.

The intuition can be seen in an example with $n = d = 2$, where c $= (2,3)$ are the coefficients in h. Suppose, initially, person 1 has $x_1 = (2,1)$ while person 2 has $x_2 = (1,2)$, so that the initial aggregate distribution is $s = \binom{7}{8}$. As a result of the unfair rearrangement, we obtain $\bar{\bar{x}}_1 = (1,1)$ and $\bar{\bar{x}}_2 = (2,2)$ and hence $\bar{\bar{s}} = \binom{5}{10}$. In other words, the unfair rearrangement of x translates into a regressive transfer from s, and hence, strictly more inequality according to the Lorenz consistency of I.

These results are summarized in the following corollary.

Corollary. A two-stage measure $M \epsilon \mathcal{M}$ satisfies anonymity, scale invariance, population replication, limited uniform majorization, and unfair rearrangement if and only if $M \epsilon \mathcal{L}$.

Each measure in $\mathcal{L}$ is determined by a vector $c \gg 0$ of coefficients and a unidimensional measure I. An approach to selecting c is given in Section V below. The choice of I can be guided by a large literature on unidimensional inequality measures. The Lorenz curve, which plays a central role in that literature, also applies directly to the present environment for fixed c. First, it provides a useful graphical depiction of the inequality in x as given by the *aggregate Lorenz curve* L_s, or the Lorenz curve applied to the aggregate vector s associated with x. Second, the resulting weak and strict Lorenz criteria can be used to rank arrays. For example, x' has strictly more multidimensional inequality than x whenever

$$L_{s'}(p) \leq L_s(p) \text{ for all } p \,\epsilon\, [0,1], \text{ with strict inequality for some } p. \quad (2)$$

Indeed, when (2) holds, it follows that every $M \in \mathcal{L}$ with the same c would agree that $M(x') > M(x)$.[28]

Given the purpose of the measure, and the desired characteristics initially posited for it, the simple, linear form of the aggregation function can be viewed as an advantage.[29] It is clearly "neutral" in the ALEP sense discussed in Kannai (1980), so that dimensional variables are neither complements nor substitutes. At the same time, the choice of c offers substantial scope for incorporating normative and practical considerations, as we shall see below. Linear aggregators often appear in empirical applications and theoretical discussions as part of a parametric family.

Following Maasoumi (1986), h typically has been interpreted as a utility function whose functional form often follows traditional examples from familiar classes. The empirical or normative bases for selecting from among the possibilities has been limited, and in any given application, several different choices for h are usually applied without identifying one, say, as a headline indicator for policy analysis. In contrast, the present paper does not adopt a utility interpretation; instead, analogous to Atkinson (1970) or Atkinson and Bourguignon (1982), it simply views h as a function employed in social evaluation. Its normative content will originate in public policy discussions of the relative importance of specific inequalities rather than from individual preference.

Finally, we should note that the results differ slightly depending on which domain is being assumed. Recall that the domain X can either be X_1 containing positive arrays or X_2 containing nonnegative arrays that have at least one positive entry per column. Domain X_1 allows a broader range of components, including those not defined for vectors with zero entries (such as many generalized entropy measures and weighted general means), but consequentially yields multidimensional measures that can be used only with positive data. Domain X_2 limits consideration to a narrower range of components defined for zero values, but then yields measures that apply more broadly.

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[28] This follows directly from the Lorenz consistency of $I \in \mathcal{I}$. Arguments entirely analogous to the proof of Theorem 2 show that for any given $c \gg 0$, the inequality quasiordering associated with the Lorenz criterion satisfies the multidimensional axioms of anonymity, scale invariance, replication invariance, limited uniform majorization and unfair rearrangement.

[29] See the discussion in Alkire and Foster (2011, p. 486).

IV. Multidimensional from Specific Inequalities

As noted above, a key desirable characteristic of an inequality measure is for it to be easily understood and communicated to others. For multidimensional measures, this could be facilitated by a clear link with specific inequality levels. In this section we show that such a link exists for many two-stage measures in $\mathcal{L}$ and for the aggregate Lorenz curve.

The key intuition is found in Shorrocks (1978), who considers the impact of the accounting period on income inequality and measuring *mobility* as the extent to which inequality falls as the accounting period is extended. Consideration is restricted to measures $I \in \mathcal{I}$ that are constant-sum convex, which includes the Gini coefficient, the generalized entropy measures, and Atkinson's family, among others.[30] Where $y_{.1}, \dots, y_{.d}$ are (column) vectors listing the incomes of n persons over d periods, Shorrocks compares the inequality in the total income $I(y_{.1} + \cdots + y_{.d})$ to the income-share weighted average of the per period income inequalities $\alpha_1 I(y_{.1}) + \cdots + \alpha_d I(y_{.d})$, where $\alpha_j = \frac{\mu(y_{.j})}{\Sigma_k \mu(y_{.k})}$ for $j = 1, \dots, d.$ He notes that the former never exceeds the latter, while the two coincide when $y_{.j}$ are scalar multiples of each other. Mobility can be defined as

$$m = \alpha_1 I(y_{.1}) + \cdots + \alpha_d I(y_{.d}) - I(y_{.1} + \cdots + y_{.d}) \geq 0 \tag{3}$$

or the extent to which total income inequality falls below the average per period income inequality due to the "smoothing" effect of aggregation across time periods.

The same logic can be applied in the multidimensional context where the smoothing now occurs across dimensions. Pick any array $x \in X$. Let $\mathcal{L}'$ denote the set of all measures in $\mathcal{L}$ with inequality components that are constant sum convex and select a measure $M \in \mathcal{L}'$ with associated components c and I. Substituting $c_j x_{.j}$ for $y_{.j}$ in equation (3) converts the final term into $I(c_1 x_{.1} + \cdots + c_d x_{.d}) = I(s) = M(x)$, or the multidimensional inequality in array x, while the first terms become the average specific inequality $A(x)$ as defined in (1) using weights

$$w_j = \frac{c_j \mu_j}{c_1 \mu_1 + \cdots + c_d \mu_d} \quad \text{for } j = 1, \dots, d \tag{4}$$

We have the following result.

[30] His formal results apply to constant-sum *strict* convex measures; implications of the weaker convexity property are discussed informally (Shorrocks 1978 p. 382).

Theorem 3. Select any $M \epsilon \mathcal{L}'$ and define $A(x)$ using (1) and (4). Then for any $x \epsilon X$ we have $m(x) = A(x) - M(x) \geq 0$, with $m(x) = 0$ if the normalized vectors $x_{\cdot j}/\mu_j$ of x are identical for $j = 1, \ldots, d$.

Proof. See the Appendix.

The theorem shows that the average specific inequality level $A(x)$ is generally larger, and certainly no smaller, than the multidimensional inequality level $M(x)$, with their difference being the Shorrocks mobility measure $m(x)$, assessed here across dimensions rather than through time. In the special case where the column vectors of x are multiples of one another (hence ordered in the same way and with identical shapes), the aggregate vector s will also be a multiple, so that $M(x) = A(x)$ and hence $m(x) = 0$.

Given any M in $\mathcal{L}'$ the mobility term can be usefully broken down into two independent terms that respectively reflect the association among dimensional distributions and their relative shapes. The *rearrangement* term $R(x) = M(\bar{\bar{x}}) - M(x)$ measures the pure effect of positive association on multidimensional inequality as one moves from x to the completely aligned version $\bar{\bar{x}}$. By Theorem 2 and the unfair rearrangement axiom we know that $R(x) \geq 0$. The *structural* term $S(x) = A(\bar{\bar{x}}) - M(\bar{\bar{x}})$ measures the mobility associated with the completely aligned vector $\bar{\bar{x}}$. It is always nonnegative by Theorem 3 given weights $w_1, \ldots, w_d$ from (4). In addition, the anonymity of I ensures that $A(x) = A(\bar{\bar{x}})$. With these observations, the next result follows immediately from Theorem 3.

Theorem 4. Select any $M \epsilon \mathcal{L}'$ and define $w_1, \ldots, w_d$ using (3). Then for any $x \epsilon X$ we have

$$M(x) = w_1 I(x_{\cdot 1}) + \cdots + w_d I(x_{\cdot d}) - R(x) - S(x) \tag{5}$$

where $\mathrm{R}(x), S(x) \geq 0$

Equation (5) provides a general expression linking multidimensional to specific inequalities, with the nonnegative terms $R(x)$ and $S(x)$ accounting for the extent to which $M(x)$ falls below $A(x)$. If $\bar{\bar{x}}$ is not a permutation of x, we know that $R(x) > 0$ by the unfair rearrangement axiom, in which case the rearrangement term is impacting measured inequality; if the columns of $\bar{\bar{x}}$ are not scalar multiples of each other, and inequality measure I is sensitive to their different shapes, then

$S(x) > 0$ by Theorem 3 and so the structural term is impacting measured inequality.[31] As an array x evolves over time, changes in multidimensional inequality can be viewed in terms of four factors: (i) changes in the specific inequality levels, (ii) changes in the weights on the specific inequalities through dimensional means, (iii) changes in positive association across dimensions, and (iv) changes in the shapes of the dimensional distributions.

The Gini coefficient G is the most common Lorenz-consistent measures, due in part to its intuitive interpretations and its clear link to the Lorenz curve. When applied to ordered vectors, the Gini becomes a linear function, which simplifies its associated two-stage multidimensional measure and expression (5) above. Let $M_G \in \mathcal{L}'$ denote the two-stage measure based on the Gini coefficient G, and let $R_G(x)$ its associated rearrangement term. We have the following expression for $M_G(x)$.

Theorem 5. Consider $M_G \in \mathcal{L}'$ and define $w_1, \dots, w_d$ using (4). Then for any $x \in X$, we have

$$M_G(x) = w_1 G(x_{\cdot 1}) + \cdots + w_d G(x_{\cdot d}) - R_G(x) \tag{6}$$

where $R_G(x) \geq 0$.

Proof. See the Appendix.

The key step of the proof uses the linear structure of G over ordered vectors to show that $S_B(x) = A_G(\bar{\bar{x}}) - M_G(\bar{\bar{x}}) = 0$, eliminating the structural term from (5).[32] Theorem 5 offers a remarkably straightforward expression for multidimensional inequality when G is used: $M_G(x)$ is the average specific Gini minus a term $R_G(x) = M_G(\bar{\bar{x}}) - M_G(x)$ that is positive for any x that is not a permutation of $\bar{\bar{x}}$, but falls to zero as a positive association in x rises towards its maximum level in $\bar{\bar{x}}$.

Suppose that a linear $h \in \mathcal{H}$ with $c \gg 0$ has been selected in the first stage. Rather than choosing a particular $I \in \mathcal{I}$ in the second stage, the Lorenz curve can be used to depict multidimensional inequality and make comparisons. Given $x \in X$ with aggregate vector s, the aggregate Lorenz curve L_s for x can be defined as $L_s(p) = \frac{1}{\mu(s)} \int_0^p Q_s(r)dr$ for $p \in [0,1]$, where $\mu(s)$ is the mean

[31] Measures with the needed sensitivity include those that are constant-sum *strictly* convex.

[32] The proof also applies to any $I \in \mathcal{I}$ that is linear over ordered vectors, such as the generalized Gini measures (Weymark 1981).

of s and $Q_s: [0,1] \to R$ is its quantile function.[33] For $j = 1, \dots, d$, let $Q_j(r)$ and $L_j(p)$ respectively denote the quantile function and Lorenz curve of the jth variable $x_{\cdot j}$ in x. Multidimensional inequality in x is evaluated using its aggregate Lorenz curve L_s while specific inequalities are evaluated using the specific Lorenz curves L_j; in each case greater inequality is indicated by a lower Lorenz curve.

Multidimensional and specific inequalities are linked for this case as well. Let L_A denote the weighted average of the specific Lorenz curves for x using the weights from (4) so that

$$L_A(p) = w_1 L_1(p) + \dots + w_d L_d(p) \quad \text{for } p \in [0,1]. \tag{7}$$

It can be shown that $L_A(p)$ is itself a Lorenz curve. Indeed, let $\bar{\bar{x}}$ be the completely aligned version of x and denote its aggregate vector by $\bar{\bar{s}}$, the associated quantile function by $Q_{\bar{\bar{s}}}$, and the aggregate Lorenz curve by $L_{\bar{\bar{s}}}$. Then

$$\begin{aligned} L_A(p) &= \frac{1}{\sum_j c_j \mu_j} \int_0^p [c_1 Q_1(r) + \dots + c_d Q_d(r)] dr \\ &= \frac{1}{\mu_{\bar{\bar{s}}}} \int_0^p Q_{\bar{\bar{s}}}(r) dr = L_{\bar{\bar{s}}}(p) \qquad \text{for } p \in [0,1] \end{aligned} \tag{8}$$

so that the average Lorenz curve L_A is identical to $L_{\bar{\bar{s}}}$, the aggregate Lorenz curve of $\bar{\bar{x}}$. This is analogous to what was found above for M_G and relies on the linearity of L in the ordered incomes of Q. Now define the *rearrangement function* $R_L: [0,1] \to R$ by $R_L(p) = L_s(p) - L_{\bar{\bar{s}}}(p)$ for $p \in [0,1]$, and note that it graphically depicts the inequality-reducing impact of dampened association in moving from $\bar{\bar{x}}$ to x.[34] We have the following result.

Theorem 6. Consider any linear $h \in \mathcal{H}$ with $c \gg 0$ and define $w_1, \dots, w_d$ using (3). Then for any $x \in X$, we have

$$L_s(p) = w_1 L_1(p) + \dots + w_d L_d(p) + R_L(p) \quad \text{for } p \in [0,1] \tag{9}$$

where $R_L(p) \geq 0$.

[33] A quantile function is a generalized inverse of the cdf of a distribution; it lists the income of each person against the percentile of the person, ranging from lowest to highest.
[34] Integrating $R_L(p)$ measures the average distance or the area between the two Lorenz curves, and hence is half of the rearrangement term for the Gini coefficient.

Proof. See the Appendix.

A Lorenz curve can be used in place of a numerical inequality measure in the two-stage approach, with L_S representing multidimensional inequality in x. Expression (9) shows how multidimensional inequality L_S can be additively decomposed into a weighted average L_A of specific Lorenz curves plus a nonnegative rearrangement function R_L reflecting the extent to which x dampens positive association in $\bar{\bar{x}}$. The final term vanishes for all p when $\bar{\bar{x}}$ is a permutation of x, but otherwise is strictly positive for some p.

V. Calibration

The above results have established the properties of these multidimensional inequality measures and explored their links with specific inequalities. We now turn to the task of implementing the measures with data. As with other measures using the joint distribution, data need to be drawn from a single source to construct the vector of achievements for each person. Dimensional variables must be viewed as cardinal variables that are comparable across people so that meaningful specific inequalities and Lorenz curves can be constructed.[35] In keeping with the stated purpose of the measure, data should be available in a base period and over time. We assume the existence of an initial array $x = x^1$ and several subsequent arrays $x^2, \dots, x^T$ for some $T \geq 2$.

Taking positive multiples of variables leaves specific inequalities unchanged but can directly impact multidimensional inequality. In empirical applications, attention is often paid to rescaling variables to make them "comparable" in some sense. For example, variables might be rescaled to a common range such as [0,1]; or normalized by dividing by the mean or another indicator of size. The resulting variables are then posited to be comparable even though no measuring rod related to inequality has been invoked. In addition, the very rescaling process that asserts comparability across dimensions can, if reapplied to each round of data, reduce comparability across time.

[35] More precisely, dimensional variables must be ratio scales; note that this fixes a natural zero value for each variable, which in the present context also has implications for comparability across variables. Related issues are discussed in Alkire and Foster (2010).

Our approach leaves variables in their original forms and accounts for the relative importance of variables, and their comparability, through the calibration of c from the aggregation function. The calibration assumes that policymakers can express the relative importance of specific inequalities using positive normative weights $\nu_1, \dots, \nu_d$ summing to 1. The weights are then applied to data in the base period to obtain a *normative* average Lorenz curve, namely

$$L_N(p) = \nu_1 L_1(p) + \cdots + \nu_d L_d(p) \tag{10}$$

where L_j for $j = 1, \dots, d$ are the specific Lorenz curves from period 1. For example, if weights are equal, L_N will be the simple average of the specific Lorenz curves; while if a weight of 0.5 is placed on income and the remaining 0.5 is equally split among the rest, the resulting L_N will more closely resemble the income Lorenz curve. The normative Lorenz curve provides a snapshot of average specific inequality in the base year using weights that reflect policy priorities.

Expression (9) linking multidimensional and specific inequalities contains a second average Lorenz curve L_A which depends on the choice of coefficients $c_1, \dots, c_d$ through its weights. The calibration approach selects these coefficients to ensure that the latter curve is the same as the former. Given the associated quantile functions $Q_j(r)$ and means μ_j for $j = 1, \dots, d,$ we can rewrite expression (10) as

$$L_N(p) = \int_0^p \frac{\nu_1}{\mu_1} Q_1(r)dr + \cdots + \int_0^p \frac{\nu_d}{\mu_d} Q_d(r)dr \tag{11}$$

which through (8) suggests the use of coefficients $c_j = \frac{\nu_j}{\mu_j}$ for $j = 1, \dots, d$. Indeed, applying these coefficients to equation (7) yields $L_A(p) = L_N(p)$ for the base period. In this way, the baseline for evaluating the evolution of inequality can reflect the policy priorities embodied in the normative weights $\nu_1, \dots, \nu_d$. An analogous relationship holds for the measure M_G associated with the Gini coefficient. Substituting $c_j = \frac{\nu_j}{\mu_j}$ into the weights $w_j = \frac{c_1 \mu_1}{\sum_j c_j \mu_j}$ used in $A_G(x)$ immediately yields $A_G(x) = \nu_1 I(x_{.1}) + \cdots + \nu_d I(x_{.d})$. So, for example, if the measure had two dimensions (say health and income) with equal normative weights, each Gini point in health would have the same value in $A_G(x)$ as a Gini point in income. Through the calibration, which accounts for both the weights and the base year means, the effective measuring rod becomes units of specific inequalities.

Once the coefficient vector c has been fixed, the measure $M \in \mathcal{L}'$ (and L_s) can be applied to distributions x^t for $t = 1, \dots, T$ for the purpose of evaluating multidimensional inequality through time. The analysis it provides is consistent with the core axioms of multidimensional inequality, while its intuitive aggregation structure helps interpret results. The link with specific inequalities likewise helps in understanding the evolution of multidimensional inequality via changes in specific inequalities, dimensional means, and joint distribution. Depending on the unidimensional inequality measure used, further analysis is possible. For example, L_s can help evaluate the robustness of trends to the choice of unidimensional measure; while the use of a decomposable I allows multidimensional inequality to be decomposed into traditional within-group and between-group terms.

VI. Illustrations.

We begin with a series of simulated results to illustrate the properties of the approach. The data generation process for our simulation is based on multivariate log-normal distributions, a class of distributions that approximates well many policy-relevant variables. We generate data for $d = 2$ outcome variables over $T = 2$ time periods by populating samples for each period with n =100,000 random draws of outcome pairs from the bivariate log-normal distribution with given means and covariance matrix.[36] In the resulting arrays x^t for $t = 1, 2$, the distribution of the first outcome is more unequal than the distribution of the second outcome. Outcomes have equal normative weights so that $c_j = \frac{1}{2\mu_j}$ for $j = 1, 2$, where μ_j is the mean of $x_{.j} = x^1_{.j}$. The resulting c is applied to x^t for $t = 1, 2$ to obtain aggregate vector s^t, while $\bar{\bar{s}}^t$ corresponds to the completely aligned version $\bar{\bar{x}}^t$ of x^t.

In Figure 1, the solid line depicts the "actual" aggregate Lorenz curve $L_{s^t}(p)$, while the dashed line depicts the completely aligned aggregate (or average) Lorenz curve $L_{\bar{\bar{s}}^t}(p)$ for each time period t = 1, 2. Figure 1 shows the simulated effect of the proportional increase (growth) in the mean of the first outcome in period 2, namely, both $L_{s^2}(p)$ and $L_{\bar{\bar{s}}^2}(p)$ shift downwards, indicating an increase in multidimensional (and average) inequality between the two periods.

[36] We use Stata command drawnorm to generate a bivariate normal distribution of two variables with the given parameters. We then exponentiate these variables to obtain the bivariate log-normal distribution (Stata 2023).

This increase is driven in part by the higher effective weight on the more unequal outcome in period 2 (via equation 7). This change also makes the array x^2 more "aligned" compared to x^1 as reflected in a lower rearrangement term in period 2, which could be seen by comparing average vertical differences between $L_{s^2}(p)$ and $L_{\bar{\bar{s}}^2}(p)$ (0.037) and $L_{s^1}(p)$ and $L_{\bar{\bar{s}}^1}(p)$ (0.045).

Figure 1: Uncorrelated outcomes, increase in the mean of the first outcome in period 2.

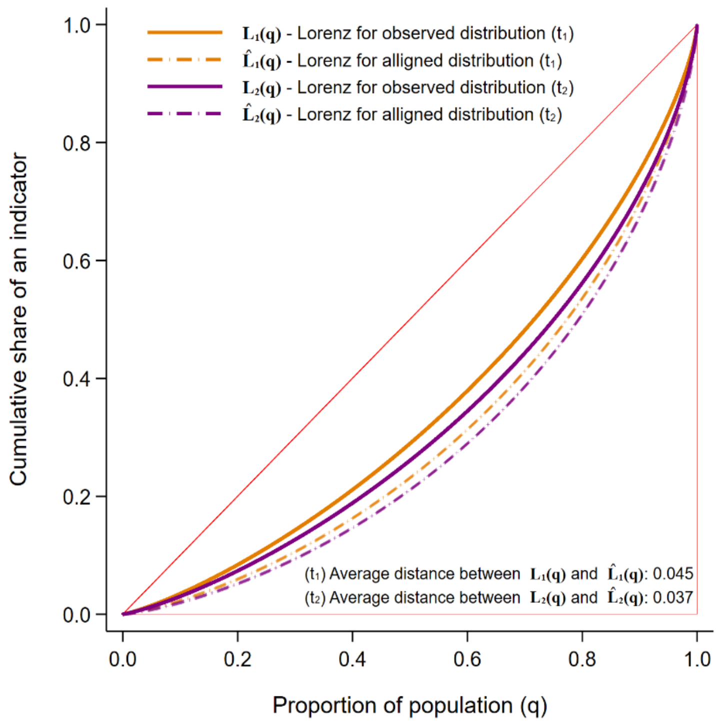


Figure 2 presents the simulation scenario where the distribution of the second outcome becomes more unequal in period 2. This change increases overall multidimensional inequality, which again is manifested by the south-east shift of $L_{s^2}(p)$ (and $L_{\bar{\bar{s}}^2}(p)$) compared to the curves in period 1. The average vertical difference between the Lorenz curves associated with the actual and the fully aligned array increases from 0.045 in period 1 to 0.051 in period 2.

Figure 2: Uncorrelated outcomes, higher inequality in the second outcome in period 2.

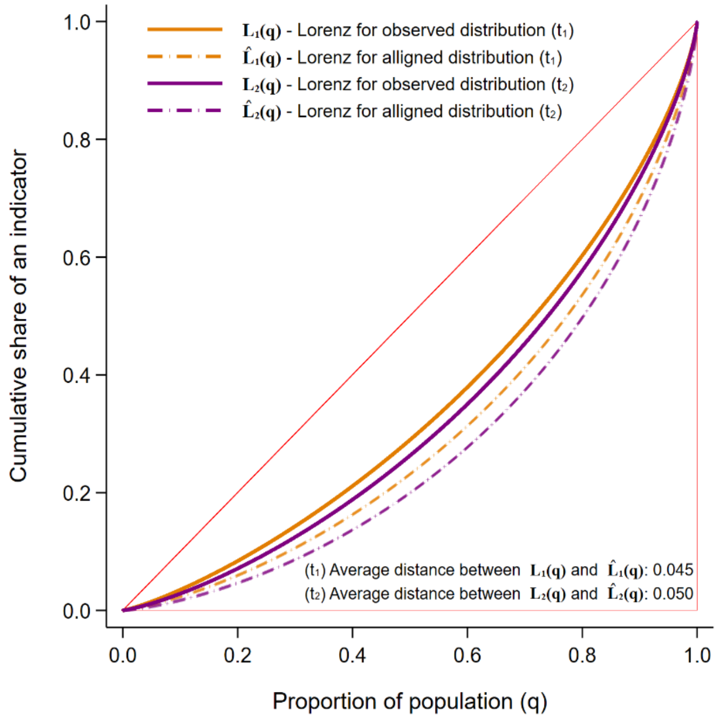
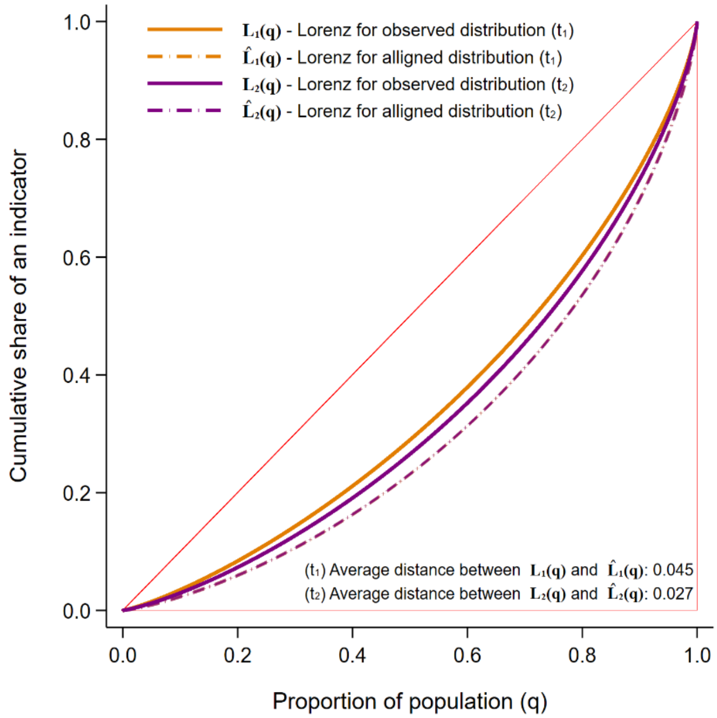


Figure 3 depicts the simulation where the two outcomes become more correlated in period 2. The higher correlation of the outcomes clearly has no effect on two fully aligned curves as $L_{\bar{\bar{s}}^1}(p) = L_{\bar{\bar{s}}^2}(p)$. But it increases the multidimensional inequality in period 2 by lowering the mobility term and pushing $L_{s^2}(p)$ towards $L_{\bar{\bar{s}}^2}(p)$, so that the average vertical difference between the actual and aligned curves decreases from 0.045 in period 1 to 0.027 in period 2.

Figure 3: Outcomes are uncorrelated in period 1 and correlated in period 2.



We now illustrate equation (5), which breaks down multidimensional inequality into average specific inequalities, and the rearrangement and structural terms, for various measures in $\mathcal{L}'$. Subtracting the structural term $S(x)$ from the average specific inequality $A(x)$ yields $M(\bar{\bar{x}})$, the multidimensional inequality of the fully aligned array. Subtracting the rearrangement term $R(x)$ from $M(\bar{\bar{x}})$ yields multidimensional inequality $M(x)$. Table 1 lists these components for the multidimensional measures generated by the Mean Log Deviation (MLD), the (first) Theil, and the Gini inequality measures for three simulation scenarios. As noted above, $S(x) = 0$ in the case of Gini, but it is positive for the other measures. For example, when the mean of the first outcome doubles, M_{MLD} increases from *0.158* to *0.212* and M_{Theil} from *0.170* to *0.235*, with much of the change being reflected in a rising $A(x)$ term and $R(x)$ and $S(x)$ falling slightly. M_{Gini} likewise increases from *0.311* to *0.360* with $A(x) = M(\bar{\bar{x}})$ rising and $R(x)$ falling by less and $S(x)$ obviously remaining unchanged.

Table 1: Changes in multidimensional inequality and components for three scenarios.[*]

	MLD		**Theil**		**Gini**	
	Period 1	Period 2	Period 1	Period 2	Period 1	Period 2
Scenario 1	*Mean income of the first outcome increases from 100 in Period 1 to 200 in Period 2*					
$M(\hat{x})$	0.270	0.323	0.281	0.335	0.400	0.434
$M(x)$	0.158	0.212	0.170	0.235	0.311	0.360
$A(x)$	0.302	0.353	0.302	0.354	0.400	0.434
$R(x)$	0.112	0.111	0.111	0.100	0.089	0.074
$S(x)$	0.032	0.030	0.021	0.019	0.000	0.000
Scenario 2	*Inequality of the second outcome increases from Gini 0.3 in Period 1 to Gini 0.4 in Period 2*					
$M(\hat{x})$	0.270	0.356	0.281	0.360	0.400	0.450
$M(x)$	0.158	0.204	0.170	0.211	0.311	0.349
$A(x)$	0.302	0.365	0.302	0.366	0.400	0.450
$R(x)$	0.112	0.152	0.111	0.149	0.089	0.101
$S(x)$	0.032	0.009	0.021	0.006	0.000	0.000
Scenario 3	*Correlation between outcomes increases from 0 in Period 1 to ≈0.3 in Period 2*					
$M(\hat{x})$	0.270	0.270	0.281	0.282	0.400	0.400
$M(x)$	0.158	0.200	0.170	0.211	0.311	0.347
$A(x)$	0.302	0.302	0.302	0.303	0.400	0.400
$R(x)$	0.112	0.070	0.111	0.071	0.089	0.053
$S(x)$	0.032	0.032	0.021	0.021	0.000	0.000

[*]) The simulated distributions have the following parameters: First outcome: mean 100, Gini 0.5; Second outcome: mean 30, Gini 0.3; Correlation: period 1: 0; period 2: ≈ 0.3.

Table 2 presents results for the three inequality measures when the outcome variables in period 2 are based on a simple transformation of the period 1 data: namely, the second variable is unchanged from period 1 while the first is obtained from its period 1 value by an additive or a proportional increase. A uniform increment of 20 units lowers multidimensional inequality, as is seen in the top panel of Table 2: M_{MLD} declines from 0.160 to 0.130, M_{Theil} declines from 0.173 to 0.144, and M_{Gini} from 0.312 to 0.284. That increment also results in a drop in the rearrangement component for all three indexes, and the structural component declines for MLD and Theil, and remains at 0 for Gini. In contrast, a proportional increase in the first outcome by 20% increases multidimensional inequality as seen in the bottom panel of Table 2. M_{MLD} grows from 0.158 to 0.171, M_{Theil} grows from 0.170 to 0.185, and M_G grows from 0.311 to 0.323, while both the rearrangement and structural mobility components change only slightly.

Table 2: Changes in multidimensional inequality from additive or proportional changes in the first outcome.*

	MLD		Theil		Gini	
	Period 1	Period 2	Period 1	Period 2	Period 1	Period 2
Scenario 1	*The first outcome variable additively increases from 100 in Period 1 to 120 in Period 2*					
$M(\hat{x})$	0.270	0.216	0.281	0.234	0.400	0.364
$M(x)$	0.158	0.129	0.170	0.142	0.311	0.283
$A(x)$	0.302	0.222	0.302	0.242	0.400	0.364
$R(x)$	0.112	0.087	0.111	0.092	0.089	0.081
$S(x)$	0.032	0.006	0.021	0.008	0.000	0.000
Scenario 2	*The first outcome variable proportionally increases from 100 in Period 1 to 120 in Period 2*					
$M(\hat{x})$	0.270	0.284	0.281	0.296	0.400	0.409
$M(x)$	0.158	0.171	0.170	0.185	0.311	0.323
$A(x)$	0.302	0.316	0.302	0.316	0.400	0.409
$R(x)$	0.112	0.113	0.111	0.111	0.089	0.086
$S(x)$	0.032	0.032	0.021	0.020	0.000	0.000

*) The simulated distributions have the following parameters: First outcome: mean 100, Gini 0.5; Second outcome: mean 30, Gini 0.3; Correlation: period 1: 0; period 2: ≈ 0.3.

We now apply the methods to analyze changes in multidimensional inequality in Azerbaijan from 2016 to 2018. We use data from the second (2016) and the fourth (2023) rounds of the Life in Transition Survey (LITS). The LITS is a survey run by the European Bank of Reconstruction and Development and the World Bank covering the so-called "transition countries" of Europe and Central Asia and several comparator countries of Western Europe, the Middle East, and North Africa (EBRD 2023). The survey included a nationally representative sample of around 1,000 households in Azerbaijan in the second and fourth rounds. Azerbaijan experienced rapid growth in per capita GDP between 2016 and 2023, which makes the example of this country helpful in illustrating the properties of the methods.

We construct the multidimensional inequality index M_{Gini} for three dimensions captured by the monthly per capita income (in 2017 PPP terms), years of education, and respondent's health assessment. The measure is calibrated for 2016 using normative weights of ½ for the income dimension and ¼ for the education and health dimensions. Table 3 presents the specific and multidimensional inequality levels for Azerbaijan in 2016 and 2023. The mean monthly per capita income increased by almost 59 percent from about 852 PPP dollars in 2016 to 1350 PPP dollars in 2023. The average years of education and health self-assessment remained relatively stable. Income growth was accompanied by an increase in income inequality, from a Gini of 0.253 in 2016 to 0.339 in 2023. The inequality in years of education grew while the inequality in health assessment slightly declined.

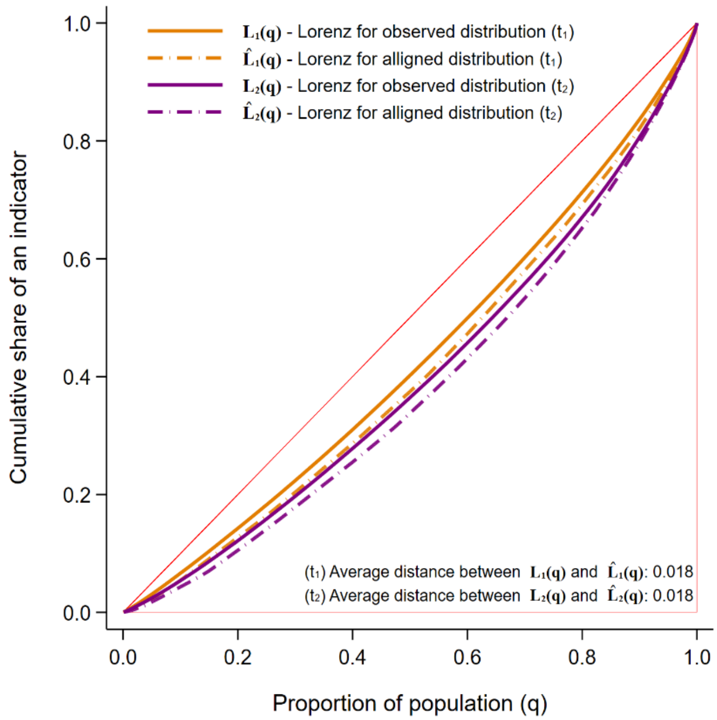
Table 3: Specific and multidimensional inequalities in Azerbaijan, 2016-2023.

	2016	**2023**
Specific Inequalities		
Income		
Mean	852.46	1384.44
Gini	0.253	0.339
Education (years)		
Mean	10.304	11.091
Gini	0.094	0.115
Health		
Mean	3.448	3.511
Gini	0.166	0.159
Multidimensional Inequality		
$A(x) = M(\bar{\bar{x}})$	0.181	0.260
$M(x)$	0.144	0.230
$m(x) = R(x)$	0.037	0.029

Multidimensional inequality as measured by M_{Gini} increased significantly between 2016 and 2023, from 0.144 to 0.230. This increase is due to (i) changes in the specific inequalities, (ii) changes in the effective weights as dimensional means change, and (iii) changes in the rearrangement term. Holding (ii) and (iii) fixed and altering specific inequalities from the 2016 to the 2023 levels raises the average specific inequality from 0.181 to 0.238. Incorporating the 2023 means alters the effective weights from 0.50, 0.25, and 0.25, respectively, to 0.61, 0.20, and 0.19, thus increasing the average further to 0.260. Finally, the rearrangement term fell slightly from 0.037 to 0.029, reflecting greater alignment of dimensions and ensuring that the increase in multidimensional inequality (namely, from 0.144 in 2016 to 0.230 in 2023) is more pronounced than the increase in average specific inequality.

Figure 4 depicts the associated Lorenz curves for 2016 and 2023. The solid curves represent $L_{s^t}(p)$, the Lorenz curve of the aggregate distribution s^t from the actual data array. The dashed curve is $L_{\bar{\bar{s}}^t}(p)$ associated with the aligned aggregate distribution $\bar{\bar{s}}^t$ or, equivalently, the weighted average of the specific Lorenz curves. The vertical difference between the solid and dashed curve is the Lorenz rearrangement function for the year, while the average vertical difference is clearly linked to the Gini rearrangement term. Each 2023 curve is below its respective 2016 curve, indicating that multidimensional inequality is unambiguously higher in 2023 than in 2016. Replacing the Gini with any Lorenz-consistent inequality index I would preserve the conclusion that multidimensional inequality rose from 2016 to 2023.

Figure 4: Multidimensional inequality in Azerbaijan.



VII. Conclusions.

The goal of this paper was to identify a technology for measuring multidimensional inequality that is axiomatically sound and can be easily understood by policymakers. The latter concern suggests the intuitive two-stage approach of Maasoumi (1986), which aggregates achievements for each person in the first stage and applies a traditional inequality measure to the distribution of aggregates in the second. Our first series of results showed that the aggregation function must take on a linear form if the measure is to satisfy the basic axioms for multidimensional inequality measures. The next set of results showed how multidimensional inequality can be expressed as a weighted average of specific inequalities minus a "mobility" term that measures the inequality-reducing impact of dimensional mixing. The mobility term can be further divided into terms reflecting the effect of rearranging achievements and combining distributions with different shapes, with the latter term disappearing when the Gini coefficient or the Lorenz curve is used in the second stage. We noted how the technology might be calibrated to incorporate the normative priorities of policymakers over the specific inequalities. A series of examples with simulated data showed how the specific inequalities, dimensional means, rearrangement term and structural term together shape multidimensional inequality, while an empirical example from Azerbaijan illustrates how multidimensional inequality evolved during a period of rapid economic growth.

There are several concepts related to multidimensional inequality that the paper does not address. We have not relied on a prior notion of welfare in our approach, and the functional form is not reliant on traditional representations of welfare, utility functions, or even preferences. While there is significant ongoing work in these areas, our paper is not intended to contribute to the associated lines of research. The capability approach provides a natural guide for expanding consideration to other evaluation spaces. As there are multiple capabilities, our multidimensional approach to inequality might be seen as moving toward the same general goal. But our approach does not account for individual variations in conversion factors between resources and outcomes, nor the value of having many choices, both of which are central to the capability approach (Foster and Sen 1997, Basu and Lopez-Calva, 2011). In addition, there is a burgeoning literature on inequality with ordinal variables.[37] In contrast, the methods presented in this paper require cardinal variables, while the link with specific inequalities also requires inequality values to have cardinal significance. Consequently, they are not applicable to the many cases where cardinality assumptions are unable to be maintained.

[37] See Allison and Foster (2004), for example.

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Appendix

Proof of Theorem 1. Let $M \epsilon \mathcal{M}$ have components h and I. Suppose that M satisfies weak uniform majorization. We begin with the following lemma.

Lemma: Suppose that $h(x_1) = \cdots = h(x_k) = \alpha > 0$ for a given set $x_1, \dots, x_k \epsilon V^{\cdots}$ of $k \geq 2$ vectors. Then $h(\lambda_1 x_1 + \cdots + \lambda_k x_k) = \alpha$ for any $\lambda_1, \dots, \lambda_k > 0$ with $\lambda_1 + \cdots + \lambda_k = 1$.

■ Let $x_1, \dots, x_k \epsilon V^{\cdots}$ satisfy $h(x_1) = \cdots = h(x_k) = \alpha > 0$, and suppose that $\lambda_1, \dots, \lambda_k > 0$ with $\lambda_1 + \cdots + \lambda_k = 1$. By linear homogeneity of h we can find $v \epsilon V^{\cdots}$ for which $h(v) = \beta$ is strictly below α. Consider the $k \times d$ arrays $x = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \epsilon X$ and $x' = \begin{bmatrix} v \\ \vdots \\ v \end{bmatrix} \epsilon X$ and define the associated $2k \times d$ array $y = \begin{bmatrix} x \\ x' \end{bmatrix} \epsilon X$. Now define a $2k \times 2k$ bistochastic matrix $Q = \begin{bmatrix} \mathrm{B} & 0 \\ 0 & \mathrm{I} \end{bmatrix}$, where B is a $k \times k$ bistochastic matrix having $(b_{11}, \dots, b_{1k}) = (\lambda_1, \dots, \lambda_k)$ as its first row, I is a $k \times k$ identity matrix, and 0 is a $k \times k$ matrix of zeros. Since M satisfies the weak uniform majorization axiom, it follows that $M(y) \geq M(y')$ for $y' = Qy$. Let s be the vector defined by $s_i = h(y_i)$ for $i = 1, \dots, 2k$ and note that $s_i = h(x_i) = \alpha$ for $i = 1, \dots, k$ and $s_i = h(v) = \beta$ for $i = k+1, \dots, 2k$. By replication invariance of I, it follows that $M(y) = I(s) = I\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. As for the vector s' associated with y', we note that $y' = \begin{bmatrix} Bx \\ x' \end{bmatrix}$, so it is immediate that $s'_i = \beta$ for $i = k+1, \dots, 2k$, while each entry s'_i for $i = 1, \dots, k$ is found by applying h to a weighted average of the rows in x, namely $s'_i = h(b_{i1} x_1 + \cdots + b_{ik} x_k)$. By the concavity of h we have $s'_i \geq b_{i1} h(x_1) + \cdots + b_{ik} h(x_k) = \alpha$ for all i. Now consider the vector s'' having the same last k entries as s', but with the first k entries in s' replaced by their mean $\gamma = \frac{1}{k}\sum_{i=1}^{k} s'_i$. By the transfer axiom and replication invariance for I we have $I(s') \geq I(s'') = I\begin{pmatrix} \gamma \\ \beta \end{pmatrix}$. Summing up, we conclude that $I\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \geq I\begin{pmatrix} \gamma \\ \beta \end{pmatrix}$ for the Lorenz consistent measure I. By construction, we know that the inequalities $\gamma \geq \alpha > \beta$ must hold; but $\gamma > \alpha$ is surely not possible, since then $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ would strictly Lorenz dominate $\begin{pmatrix} \gamma \\ \beta \end{pmatrix}$ leading to $I\begin{pmatrix} \alpha \\ \beta \end{pmatrix} < I\begin{pmatrix} \gamma \\ \beta \end{pmatrix}$. Consequently, $\gamma = \alpha$, and since $\alpha = \frac{1}{k}\sum_{i=1}^{k} s'_i$ with $s'_i \geq \alpha$ for all i, it follows that $s'_i = \alpha$ for all i. In particular, setting $i = 1$ yields $h(\lambda_1 x_1 + \cdots + \lambda_k x_k) = h(b_{11} x_1 + \cdots + b_{1k} x_k) = s'_i = \alpha$, as desired. ■

Now consider the collection of vectors $f_1, \dots, f_d \in V^{\cdots}$ defined by $f_j = \frac{v_j}{h(v_j)}$, where $v_1, \dots, v_d \in V^{\cdots}$ are constructed from the usual basis vectors $e_1, \dots, e_d$ and the midpoint $m = \left(\frac{1}{d}, \dots, \frac{1}{d}\right)$ as follows: $v_j = \frac{1}{2}e_j + \frac{1}{2}m$ for $j = 1, \dots, d$. It can be shown that $f_1, \dots, f_d$ are linearly independent, and hence that the $d \times d$ matrix $F = \begin{bmatrix} f_1 \\ \vdots \\ f_d \end{bmatrix}$ is invertible. Define $c = (c_1, \dots, c_d) \in R^d$ by $c = F^{-1}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and note that $cf_j = h(f_j) = 1$ for all j. The vectors $f_1, \dots, f_d$ generate a simplex

$$S = \{v \in V^{\cdots}: v = \alpha_1 f_1 + \cdots + \alpha_d f_d \text{ for } \alpha_1, \dots, \alpha_d > 0 \text{ with } \alpha_1 + \cdots + \alpha_d = 1\}$$

and a cone

$$C = \{v \in V^{\cdots}: v = \beta_1 f_1 + \cdots + \beta_d f_d \text{ for } \beta_1, \dots, \beta_d > 0\}.$$

We will show that $h(v) = cv$ for all $v \in C$. First, pick any $v \in S$. By the definition of v as a point in S, it follows that $cv = \alpha_1 cf_1 + \cdots + \alpha_d cf_d = 1$; applying the lemma ensures that $h(v) = 1$. Consequently $h(v) = cv$ for all $v \in S$. Now pick any $v \in C$. By the definition of v as a point in C, it follows that $cv = \beta_1 cf_1 + \cdots + \beta_d cf_d = \beta_1 + \cdots + \beta_d$, so that $v' = v/(\beta_1 + \cdots + \beta_d) \in S$. Consequently, by the linear homogeneity of h we have $h(v) = (\beta_1 + \cdots + \beta_d)h(v') = (\beta_1 + \cdots + \beta_d) = cv$, which shows that $h(v) = cv$ for all $v \in C$.

We now show that $h(v) = cv$ for any $v \in V^{\cdots}$ that is not in C. Pick any such v and define $v' = v/h(v)$ so that by linear homogeneity $h(v') = 1$. Define $m_f = (f_1 + \cdots + f_d)/d \in S$ and note that $h(m_f) = cm_f = 1$. Since m_f is interior to C, we can find a small enough weight $0 < \lambda < 1$ so that $v'' = \lambda v' + (1-\lambda)m_f \in C$. By the lemma, $h(v'') = 1$ and hence $cv'' = 1$ since $v'' \in C$. Clearly, $cv'' = \lambda cv' + (1-\lambda)cm_f$, or equivalently $1 = \lambda cv' + (1-\lambda)$, which yields $\lambda = \lambda cv'$ and so $1 = cv' = cv/h(v)$. It follows, then, that $h(v) = cv$ for $v \in V^{\cdots}$. Finally, since h is increasing, it we know that $c \gg 0$.

Proof of Theorem 2. Let $M \in \mathcal{L}$ be any two-stage measure with components $h \in \mathcal{H}$ and $I \in \mathcal{I}$. To show that M satisfies anonymity, let x' be obtained from x by a permutation, so that $x' = \Pi x$ for some permutation matrix Π. Clearly $s' = \Pi s$, and hence $I(s') = I(s)$ by anonymity of I, which yields $M(x') = M(x)$, as required. To show that M satisfies scale invariance, let x' be obtained from x by a scalar multiple, so that $x'_{ij} = \alpha x_{ij}$ for some $\alpha > 0$, for all $i = 1, \dots, n$ and $j = 1, \dots, d$. Clearly $s' = \alpha s$ by linear homogeneity of h, and hence $I(s') = I(s)$ by the scale invariance of I, which yields $M(x') = M(x)$, as required. To show replication invariance, let x' be obtained from x by a replication. Clearly s' is obtained from s by a replication, and hence $I(s') = I(s)$ by the replication invariance of I, which yields $M(x') = M(x)$, as required.

Now to show that that M satisfies limited uniform majorization, let x' be obtained from x by a uniform majorization, so that $x' = \mathrm{B}x$ for some bistochastic matrix B. Clearly $s'_i = h(b_{i1}x_1 + \dots + b_{in}x_n) = b_{i1}h(x_1) + \dots + b_{in}h(x_n) = b_{i1}s_1 + \dots + b_{in}s_n$ and so $s' = Bs$. Hence $I(s') \leq I(s)$ by the transfer axiom for I, which yields $M(x') \leq M(x)$, as required by the first part of the axiom. Now suppose that x' is obtained from x by a uniform smoothing, and that in addition, s' is not obtained from s by a permutation. By the above we know that $s' = Bs$, and given that s' is not obtained from s by a permutation, it follows from the transfer axiom that $I(s') < I(s)$, and hence $M(x') < M(x)$, as required by the second part of the axiom.

Finally, to show that that M satisfies the unfair rearrangement axiom, let x' be obtained from x by an unfair rearrangement, so that $x' = \bar{\bar{x}}$. We first show that $M(\bar{\bar{x}}) \geq M(x)$, or equivalently $I(\bar{\bar{s}}) \geq I(s)$. Consider the ordered vector $\hat{s}$ of s and similarly permute the rows of x to obtain an array x'' whose aggregate vector is $\hat{s}$. Note that $\bar{\bar{x}}$ is also an unfair rearrangement of x'', which implies for each $j = 1, \dots, d$ that the column vector $\hat{x}_{.j}$ of $\bar{\bar{x}}$ contains the same entries as the column vector $x''_{.j}$ of x''. And since $\hat{x}_{.j}$ is ordered from lowest to highest, it follows that

$$\sum_{i=1}^{k} \hat{x}_{ij} \leq \sum_{i=1}^{k} x''_{ij} \text{ for } k = 1, \dots, n, \text{ with equality holding for } k = n \qquad \text{(A1)}$$

Multiplying through by c_j and summing across all j yields

$$\sum_{i=1}^{k} \sum_{j=1}^{d} c_j \hat{x}_{ij} \leq \sum_{i=1}^{k} \sum_{j=1}^{d} c_j x''_{ij} \text{ for } k = 1, \dots, n, \text{ with equality holding for } k = n.$$

Note that $\bar{\bar{s}}_i = \sum_{j=1}^{d} c_j \hat{x}_{ij}$ while $\hat{s}_i = \sum_{j=1}^{d} c_j x''_{ij}$ and hence

$$\sum_{i=1}^{k} \bar{\bar{s}}_i \leq \sum_{i=1}^{k} \hat{s}_i \quad k = 1, \dots, n, \text{ with equality holding for } k = n. \qquad \text{(A2)}$$

Since each column vector $\hat{x}_{.j}$ of $\bar{\bar{x}}$ is ordered, the aggregate vector $\bar{\bar{s}}$ is ordered. Consequently, (A2) is equivalent to the statement that s weakly Lorenz dominates $\bar{\bar{s}}$. It follows that $I(\bar{\bar{s}}) \geq I(s)$ for any Lorenz consistent I, and so $M(\bar{\bar{x}}) \geq M(x)$.

Now suppose that, in addition, $\bar{\bar{x}}$ is not a permutation of x. We need to verify that $M(\bar{\bar{x}}) > M(x)$. To do this we will show that at least one of the inequalities in (A2) is strict, in which case s would strictly Lorenz dominate $\bar{\bar{s}}$, implying $I(\bar{\bar{s}}) > I(s)$ for Lorenz consistent I and yielding the desired conclusion. So suppose, by way of contradiction, that all the inequalities in (A2) hold with equality. Since $\bar{\bar{s}}$ and $\hat{s}$ are ordered vectors, this would imply that that $\bar{\bar{s}} = \hat{s}$. Inequality (A1) applied to $k = 1$ for $j = 1, \dots d$ implies that $\bar{\bar{x}}_1 \leq x_1''$. Since $\bar{\bar{s}}_1 = s_1''$, the vector dominance cannot be strict, and so $\bar{\bar{x}}_1 = x_1''$; both $\bar{\bar{x}}$ and x'' have the same first row. Now suppose that the first $k' - 1$ rows in $\bar{\bar{x}}$ are the same as the first $k' - 1$ rows in x'', where $k' = 2, \dots, n$. We want to show that row k' is the same for both. By (A1) applied to k' for $j = 1, \dots, d$ we know that $\bar{\bar{x}}_{k'} \leq x_{k'}''$, since the first $k' - 1$ terms in each summation are identical. And since $\bar{\bar{s}}_{k'} = s_{k'}''$, the vector dominance cannot be strict, and so $\bar{\bar{x}}_{k'} = x_{k'}''$; both $\bar{\bar{x}}$ and x'' have the same first k' rows. This leads to the conclusion that $\bar{\bar{x}} = x''$ must be a permutation of x, contrary to assumption. Thus, there must be a strict inequality in (A2) for some $k = 1, \dots, n - 1$, from which it follows that $M(\bar{\bar{x}}) > M(x)$.

Proof of Theorem 3. Let $M \in \mathcal{L}'$ and pick any $x \in X$. Let s be the aggregate vector associated with x, and notice that $\mu(s) = c_1\mu_1 + \cdots + c_d\mu_d$. Define $s' = s/\mu(s)$ and $x'_{\cdot j} = x_{\cdot j}/\mu_j$ for $j = 1, \dots, d$, which share the same population size and same total (or mean). By constant-sum convexity, $I(s') \leq w_1 I(x'_{\cdot 1}) + \cdots + w_d I(x'_{\cdot d})$, where $w_1, \dots w_d$ defined by $w_j = c_j\mu_j/\mu(s)$ are nonnegative and sum to 1. By anonymity of I we then have $I(s) \leq w_1 I(x_{\cdot 1}) + \cdots + w_d I(x_{\cdot d})$ and hence $m(x) = A(x) - M(x) \geq 0$. If $x'_{\cdot j}$ are all identical, then so is their convex combination s' and hence $m(x) = A(x) - M(x) = 0$ in this case.

Proof of Theorem 5. Pick any $x \in X$. It will be shown that $S(x) = A_G(\bar{x}) - M_G(\bar{x}) = 0$ which along with Theorem 4 will establish the result. Note that the Gini coefficient $G: V^{\vdots} \to R$ can be defined for vector $v \in V^{\vdots}$ by $G(v) = \frac{1}{\mu(v)} \sum_{i=1}^{n} a_i \hat{v}_i$ where $a_i = \left(\frac{2i-n-1}{n^2}\right)$. Consequently,

$$A_G(\bar{x}) = \frac{c_1\mu_1}{\sum_j c_j\mu_j} G(\hat{x}_{\cdot 1}) + \cdots + \frac{c_d\mu_d}{\sum_j c_j\mu_j} G(\hat{x}_{\cdot d}) = \frac{c_1}{\sum_j c_j\mu_j} \sum_{i=1}^{n} a_i \hat{x}_{i1} + \cdots + \frac{c_d}{\sum_j c_j\mu_j} \sum_{i=1}^{n} a_i \hat{x}_{id}.$$

Let $\bar{s} \in V^{\vdots}$ be the aggregate vector associated with $\bar{x}$. Then $\bar{s}_i = c_1\hat{x}_{i1} + \cdots + c_d\hat{x}_{id}$, and $\mu(\bar{s}) = \sum_j c_j\mu_j$ so that $A_G(\bar{x}) = \frac{1}{\mu(\bar{s})} \sum_{i=1}^{n} a_i \bar{s}_i$. Since $\bar{s}$ is the aggregate vector for $\bar{x}$, it follows that

$M_G(\bar{\bar{x}}) = G(\bar{\bar{s}})$; and since the rows of $\bar{\bar{x}}$ are ordered by vector dominance, $\bar{\bar{s}}$ itself is an ordered vector, and hence $G(\bar{\bar{s}}) = \frac{1}{\mu(\bar{\bar{s}})} \sum_{i=1}^{n} a_i \bar{\bar{s}}_i$. It follows that $M_G(\bar{\bar{x}}) = A_G(\bar{\bar{x}})$, as desired.

Proof of Theorem 6. Let $h \in \mathcal{H}$ be linear with $c \gg 0$, and let L_s be the aggregate Lorenz curve for $x \in X$. Equation (8) showed that $L_A(p) = L_{\bar{\bar{s}}}(p)$ and so $L_A(p) + R_L(p) = L_s(p)$ for $p \in [0,1]$ as required by (9). By expression (A1) in the proof of Theorem 2, we know that $R_L(p) = L_s(p) - L_{\bar{\bar{s}}}(p) \geq 0$ for $p \in [0,1]$. If in addition $\bar{\bar{x}}$ is a permutation of x, then $\bar{\bar{s}}$ is a permutation of s and hence their quantile functions and Lorenz curves are identical, which implies $m_L(p) = 0$ for all $p \in [0,1]$. If $\bar{\bar{x}}$ is a not permutation of x, then the proof of Theorem 2 shows that s Lorenz dominates $\bar{\bar{s}}$ and hence $m_L(p) \neq 0$ for some $p \in [0,1]$.