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COMPETITIVE AND NONCOMPETITIVE DEMAND STRUCTURES IN LINEAR PROGRAMMING MODELS

by

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COMPETITIVE AND NONCOMPETITIVE DEMAND STRUCTURES IN LINEAR PROGRAMMING MODELS*

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Abstract

This paper developes structures for multiproduct demand sets which permit determination of alternative market forms in linear optimization models. In these structures, market equilibrium prices become endogenous, as functions of the primal solution. The approach permits constraints on a measure of income which is evaluated at the endogenous prices. Some properties of the linearized demand set are proven. Numerical solutions are obtained with linear programming via approximation techniques in convex programming; the techniques permit arbitrarily close approximation to non-linear forms without increasing the number of rows in the linear program. The sane structures are applicable to factor markets..

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1. Introduction

In the classic linear programming models of an economy or sub-economy, goods are assumed either to face infinitely elastic demand functions or to be traded in bounded quantities. Modifications sometimes are made for exports which constitute a significant share of the world market, leading to a formulation in which the optimizing unit equates marginal revenue and marginal cost on the export markets. However, the existence of international markets frequently is used to justify the price-taker assumption. But, for a large class of products, particularly agricultural commodities, the spread between c.i.f. and f.o.b. prices may be twenty percent or more, and for another group of products trading opportunities effectively do not exist. In these cases, domestic product demand functions are relevant in price determination.

Incorporating product demand functions into a planning model designed for the purpose of analyzing policy alternatives, rather than assuming exogenously determined product prices, has three principal advantages: First, it allows the model to correspond to a market equilibrium. The effects of various policies, e.g., subsidizing or taxing product or input prices, or varying the exchange rate, can then be investigated. Second, it allows the model greater flexibility. For instance, substitution between capital and labor, corresponding to different factor price ratios, can occur not only directly through the technology set or through changes in the commodity mix of output, but also through substitution in demand due to changing relative prices of products which are more or less labor or capital intensive. Third, it permits an appraisal of the distribution between consumers and producers of benefits acruing from changes in output. For example, in the common situation of agricultural production for the domestic market In the face of price-inelastic demand, the returns from increased output are negative to producers as a whole and positive to consumers. [1/]

Competitive equilibrium under downward-sloping product demand curves can be attained via maximization of the sum of producer and consumer surpluses. This is the tradition of Samuelson (1952) and Takayama and Judge (1364, 1971). The monopolistic case is one of net revenue maximization. This paper develops approximation methods for handling either form of market equilibruim in a linear programming planning model. The procedure has particular application to sectoral planning models, and one such application is given. The exposition is in the context of a static model (although it can readiiy be extended to a dynamic model).

?. Competitive and Noncompetitive Product Markets under Optimization

The specification of the objective function follows from the hoice of market form to be incorporated in the model. In the competitive [illegible] act as price takers ana **equate** marginal costs to the prices [illegible] products. in the monopolistic case, the sector maximizes its net income [illegible] marginal costs to the marginal revenues of prcducts.

[illegible] implicity of exposition, it is assumed that no cxtcrnnl [illegible]. Import-export opportunities can readily be introduced.

[illegible] these advantages obtain when a model is specified with fixed [illegible] targets and marginal supply prices for products are derived from the dual solution.

The set of domestic narket demand functions is written, assuming linearity, as

$$p = a + Bq \quad \ldots\ldots (1)$$

where p is a C x 1 vector of prices

a is a C x 1 vector of constants

B is a C x C negative definite matrix of demand coefficients

q is a C x C vector of quantities

Defining c(q) as a C x 1 vector of <u>total</u> cost functions, 1/ the objective function for the competitive case becomes:

$$Z = q' (a + .5\, Bq) - c(q) \quad \ldots\ldots (2)$$

where the first term of the RHS is the area under the demand function, i.e., the integral of (1) w.r.t. the vector q. Setting the derivative of (2) w.r. t. q equal to zero yields:

$$p = \frac{d\, c(q)}{dq} \quad \ldots\ldots (3)$$

which is the equilibrium condition of price equals marginal cost.

The objective function, Z, can be decomposed into components which correspond to consumer surplus and producer surplus: 2/

$$CS = .5\, q' (a - p) = - .5\, q' Bq \quad \ldots\ldots (4)$$

$$PS = q'p - c(q) = q' (a + Bq) - c(q) \quad \ldots\ldots (5)$$

1/ The supply functions of a programming model usually arc endogenous, but that does not affect the generality of this exposition for the product demand side.

2/ Of course, the function (2) may be interpreted merely as an equilibrium-seeking device, thus side-stepping the controversies sorrounding the Marshallian surpluses. (See, for example, Mishan (1968).) However, if it is accepted as a social utility function, some interesting programming experiments are possible, as outlined later in this paper. An alternative interpretation of the objective function is possible; it can be interpreted as the profit function of a discriminating monopolist. Such an interpretation, of course, is hardly tenable for a sector planning modei, partly on account of problems of separability of markets, but also because of the fact that the demand functions would require some re-formulation on account of income effects.

For any differentiable set of demand functions a similar result holds: the appropriate maximand for the competitive equilibrium solution is the sum of producer and consumer surpluses. [1/] Of course, in mathematical programming formulations, maximization is generally constrained.

The appropriate objective for the monopolistic case is: [2/]

$$Y = q' (a + Bq) - c(q) \quad \ldots\ldots (6)$$

which yields the equilibrium condition that

$$a + 2\,Bq = \frac{d\ c(q)}{dq} \quad \ldots\ldots (7)$$

where the left-hand term is a vector of marginal revenues.

In both cases, the maximand involves a quadratic form in q. Problems arise in practice because non-linear programming models rapidly approach the bounds of computer technology as the models become large. For this reason, approximation procedures were sought in order to take advantage of the computational efficiencies of linear programming. Two such procedures have been developed; the first for the case where estimates of the coefficients of B are available, and the second where less information is to be had on the structure of demand interrelationships.

1/ This objective Function is essentially identical to Samuelson's "net social payoff" function, except that he includes interregional transportation costs whereas here only a single point in space is treated. See Samuelson (1952). The same objective Function is elaborated in the multi-product case by Takayama and Judge (1964). Possible applications to agriculture have been discussed by Yaron, Plessner, and Heady ([illegible]) and an actual application (in a different form than presented in this paper) was made by Farhi and Vercueil (1969). An application to achieve competitive equilibrium in the market for irrigation water was made by Guise and Flinn (1970).

2/ Plessner (1971) treats this maximand with quadratic programming. He also develops structures for the "leading firm" market form.

In this exposition, it is assumed that costs are accounted for in the factor-supply activities of a programming model, so that the demand activities account only for the areas under the demand function (in the competitive case) or the area under the marginal revenue function (in the monopolistic case). For one product, in the competitive case, this area is:

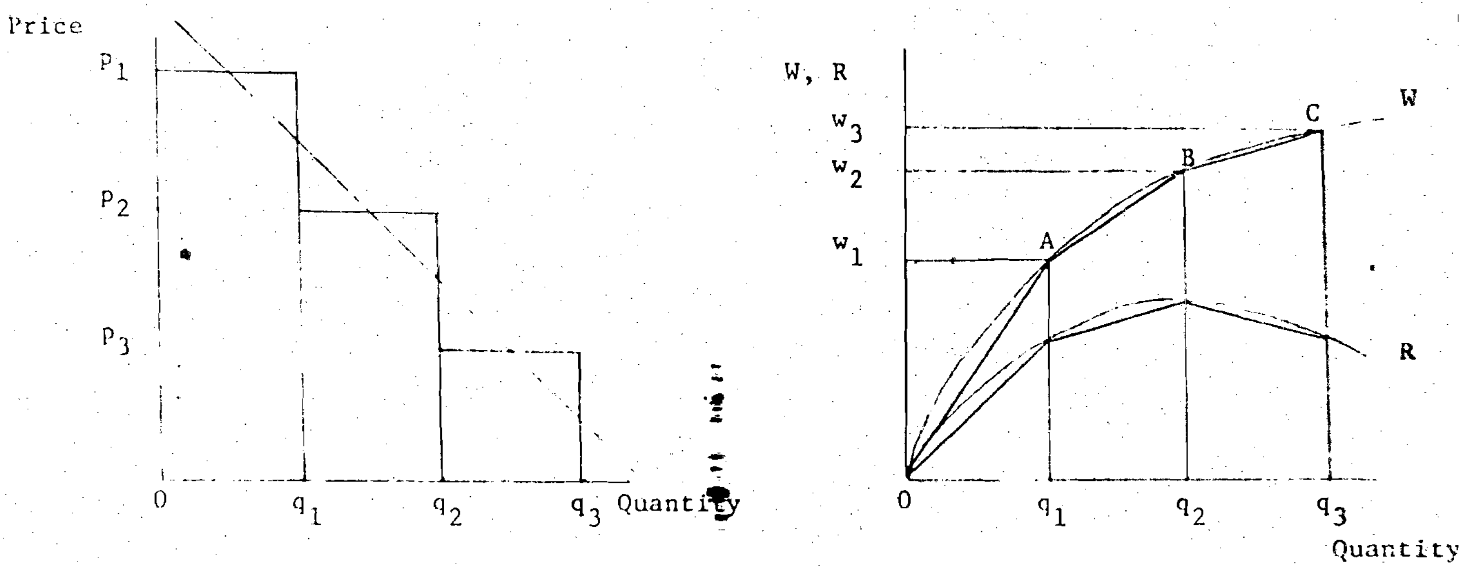
$$W = q\ (a + .5\ Bq) \quad \ldots\ldots (8)$$

and the corresponding revenue function is

$$R = q\ (a + Bq) \quad \ldots\ldots (9)$$

These are the functions, together with the corresponding demand functions, sketched in Figure 1, assuming only three segments in the approximation.

Figure 1



The convexity of either function follows from the convexity of a well-behaved demand set. This in turn may be demonstrated by reference to the properties of the Hessian matrix of detached coeificients of the joint demand functions. For linear demand functions, W and R are nonlinear. However, direct segmentation of W (or R), rather than the demand function, permits arbitrarily close approximation to the nonlinearity in a linear programming nodei, without increasing the number of rows in the model. 1/ This is shown in the following section.

3. The LP Tableau for the Separable Case

In the single-product case, the linear programing tableau corresponding to the segmented approximation of the functions W and R for *one* product is the following, taking equation (8) as the maximand and equation (9) as an income constraint, or income accounting equation:

Tableau I

	Production activities				Selling activities					RHS
Objective function	$-c_1$	$-c_2$	...	$-c_m$	w_1	w_2	w_3	...	w_S	(Max)
Income constraint	$-c_1$	$-c_2$	...	$-c_m$	r_1	r_2	r_3	...	r_S	$\geq Y^*$
Commodity balance	y_1	y_2	...	y_m	$-q_1$	$-q_2$	$-q_3$	...	$-q_S$	≥ 0
Demand constraint					1	1	1	...	1	≤ 1

where c_i are costs associated with the production activities;

y_i are physical outputs of the production activities at unit level;

w_s are values of W corresponding to q_s.

1/ This is an application of the grid-linearization technique of separable programming. See C. Miller (1963) and G. Hadley (1964).

r_s are values of R corresponding to q_s;

q_s are the total quantities sold at the limit of each segment of the function W;

$Y^* \geq 0$ is the target level of producer's incomes, which is set equal to zero (or at any non-negative value such that the constraint is non-binding), for the competitive case. It becomes the objective function in the monopolistic case; and

s = 1, 2, ..., S is the segment index.

Note that in Tableau 1, no more than two activities from the set of selling activities (each corresponding to one segment in the approximation) will enter the optimal basis at positive levels. This may be seen by reference to the W-function in figure 1: a linear combination of more than two points is a line interior to the piece-wise efficiency frontier OABC.

Tableau 1 is a transformaticn (using elementary row and column operations) of an initial tableau which embodies additively separable segments with a separate bound for each segment. This initial tableau is as follows:

Tableau 2

	Production activities			Selling activities				RHS
Objective function	$-c_1$	$-c_2$ ...	$-c_m$	p_1	p_2	p_3 ...	p_S	(Max)
Income constraint	$-c_1$	$-c_2$ ...	$-c_m$	m_1	m_2	m_3 ...	m_S	$\geq Y^*$
Commodity balance	y_1	y_2	y_m	-1	-1	-1 ...	-1	≤ 0
Demand segment constraints				1				$\leq K_1$
					1			$\leq K_2$
						1		$\leq K_3$
								..
							1	$\leq k_S$

where p_s and m_s are prices and marginal revenues corresponding to segments s of the demand function. It will be noted that Tableau 2 corresponds to segmenting the demand and marginal revenue functions as step function;, rather than to the linearization of the W and R functions which underlies Tableau 1. 1/

The principal advantage of Tableau 1 over Tableau 2 is that the demand function (or area function W) can be approximated as closely as desired without requiring additional constraints in the program. The number cf selling activities increases as the number of linear segments increases, but the number of rows remains constant.

The approach is readily extended to two or more products which are additively separable in demand, with one commodity balance per product and one convex combination constraint per product. For example, with the first commodity's W-function being segmented into two parts only, corresponding to q_{11} and q_{12}, and the second commodity's W-function being segmented into three parts, corresponding to q_{21}, q_{22}, and q_{23}, the approximation in . sample linear program takes the form of the following tableau:

Table 3

	Production activities		Selling activities					RHS
	Good 1	Good 2						
Objective function	$-c_{1j}$	$-c_{2j}$	w_{11}	w_{12}	w_{21}	w_{22}	w_{23}	(Max)
Income constraint	$-c_{1j}$	$-c_{2j}$	r_{11}	r_{12}	r_{21}	r_{22}	r_{23}	$\geq Y^*$
Commodity balance 1	y_{ij}		$-q_{11}$	$-q_{12}$				≥ 0
Commodity balance 2		y_{2j}			$-q_{21}$	$-q_{22}$	$-q_{23}$	≥ 0
Demand constraint 1			1	1				≤ 1
Demand constraint 2					1	1	1	≤ 1

1/ To the extent that non-linear functions are incorporated into planning models, their inclusion as step functions is a common procedure. For a recent example, see MacEwan (1971) pp 66-69.

where c_{ij} are costs for the i^{th} product in the j^{th} activity producing it;

y_{ij} are unit outputs of the ith prodcct in the j^{th} activity producing it;

q_{ij} define the quantities of the i^{th} product. corresponding to the j^{th} segment;

w_{ij} are values of W corresponding to the j^{th} level of the i^{th} commodity; and

r_{ij} are the values of R corresponding to the j^{th} level of the i^{th} commodity.

4. Substitution in Demand

In the event that two or more products are not separable in demand, the nonlinear demand set can be linearized directly, to an arbitrarily close approxisation, by specification of activity vectors representing points on the demand surface and by incorporating an appropriate convex combination constraint. An example of the tableau in such a case, for two products and six segments per term in the objective functiun, is as follows:

Table 4

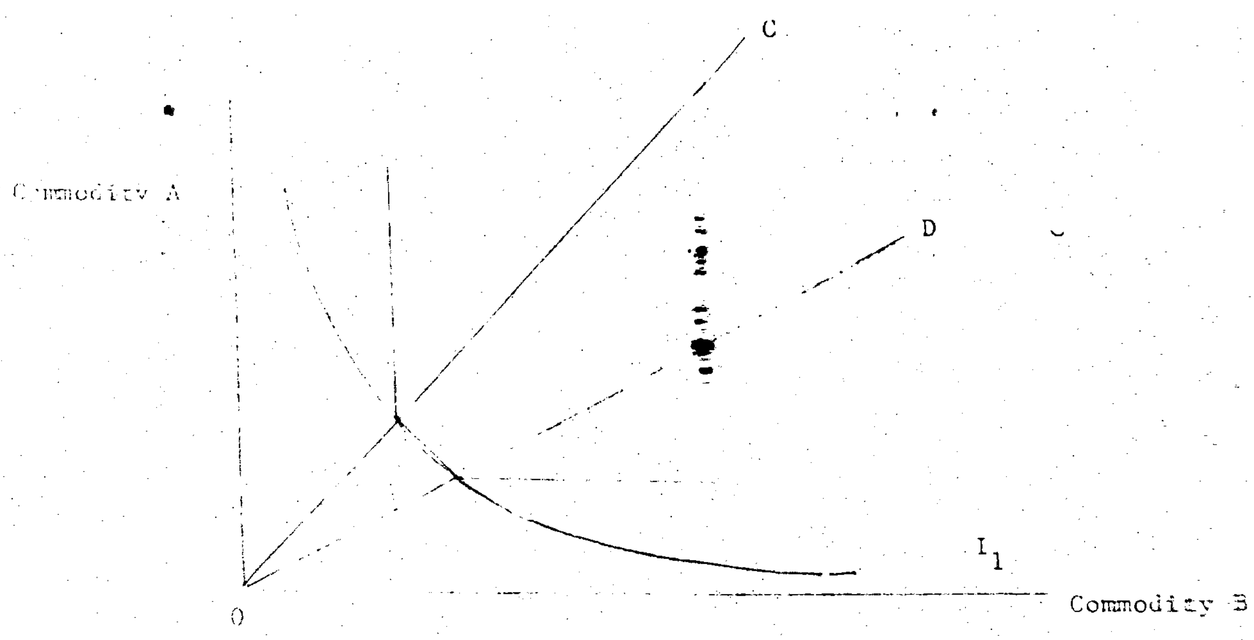
	Production activities Good 1	Production activities Good 2	Selling activities						RHS
Objective function	$-c_{1j}$	$-c_{2j}$	w_{11}	w_{12}	w_{13}	w_{21}	w_{22}	w_{23}	(Max)
Income constraint	$-c_{1j}$	$-c_{2j}$	r_{11}	r_{12}	r_{13}	r_{21}	r_{22}	r_{23}	$\geq Y^*$
Commodity balance 1	y_{1j}		$-q_{11}$	$-q_{11}$	$-q_{11}$	$-q_{12}$	$-q_{12}$	$-q_{12}$	≥ 0
Commodity balance 2		y_{2j}	$-q_{21}$	$-q_{22}$	$-q_{23}$	$-q_{21}$	$-q_{22}$	$-q_{23}$	3 0
Convex combination constrnint			1	1	1	1	1	1	6 1

where the symbols are defined as in Tableau 3.

In the above treatment, it is assumed that the elements of the matrix B, including off-diagonal elements, in equation (1) are known or can be estimated. Frequently, the available information consists only of estimates of own-price elasticities for a number of individual commodities and commodity groups.

The basis of the approximation procedure developed for this situation of limited information is the assumption that commodities can be classified into groups, such that the marginal rate of substitution (MRS) is zero between all groups but non-zero and constant within each group. Clearly this assumption is only an approximation to reality. A group may consist of one or more commodities, and limits are defined on the variability of the commodity mix within each group. The relevant portions of the indifference surface with respect to two commodities in a group are shown in figure 2. The rays OC and OD in the figure define the limits on the composition of the commodity bundle.

Figure 2



If sufficient information is available the approach can be extended to more linear segments per indifference curve, each segment representing a different value of the MRS.

Consider a group consisting of C commodities. The appropriate LP tableau may be represented as follows:

Tableau 5

	Production activities	Selling activities	RHS
Objective function	- C	$W_1 \ldots W_s \ldots W_S \ldots$	(Max)
Income constraint	-C	$R_1 \ldots R_s \ldots R_S \ldots$	$\geqslant Y^*$
Commodity balances	Y	$-Q_1 \ldots -Q_s \ldots -Q_S \ldots$	$\geqslant 0$
Convex combination constraint		$\underset{\sim}{1} \ldots \underset{\sim}{1} \ldots \underset{\sim}{1} \ldots$	$\leqslant 1$

where s = 1, 2, ..., S is a segment index;

C is a row vector of production costs;

Y is a C–rowed matrix of production coefficients entering the commodity balances;

W_s, R_s are 1 x C vectors of areas under the demand function and gross revonues respectively; and

Q_s is a C x C matrix of adjusted quantities as defined below.

In Tableau 5, each of the block of activities $\begin{bmatrix} W'_s & R_s & -Q_s & \underset{\sim}{1}' \end{bmatrix}$ constitutes a set of "nixing" activities for one segment of the composite demand function for the commodity group. This biock of activities can be written as:

$$\begin{bmatrix} W_s \\ R_s \\ -Q_s \\ 1 \end{bmatrix} = \begin{bmatrix} w_s & w_s & \cdots & w_s & \cdots & w_s \\ r_s & r_s & \cdots & r_s & \cdots & r_s \\ -q_{s11} & -q_{s12} & \cdots & -q_{s1m} & \cdots & -q_{s1M} \\ -q_{s21} & -q_{s22} & \cdots & -q_{s2m} & \cdots & -q_{s2M} \\ -q_{sc1} & -q_{sc2} & \cdots & -q_{scm} & \cdots & -q_{scM} \\ \vdots & \vdots & & \vdots & & \vdots \\ -q_{sC1} & -q_{sC2} & \cdots & -q_{sCm} & \cdots & -q_{sCM} \end{bmatrix} \quad \ldots\ldots (10)$$

where the elements are as defined below.

The derivation of formulae for the elements of (10) is tedious, because they take account of shifts both between and among segments. The starting point is a set of observed prices $\{\bar{p}_1, \ldots, \bar{p}_c, \ldots, \bar{p}_C\}$ and of quantities $\{\bar{q}_1, \ldots, \bar{q}_c, \ldots, \bar{q}_C\}$. Relative prices of commodities in the group are assumed fixed, both within and betweensegments, and are defined by

$$\rho_c = \bar{p}_c \,/\, \textstyle\sum_c \bar{p}_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Corresponding to the observed sets of prices and quantities are a quantity index,

$$\bar{q} = \textstyle\sum_c \bar{q}_c \, \bar{\rho}_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

and a price index,,

$$\bar{q} = \textstyle\sum_c \bar{p}_c \, \bar{q}_c / \bar{V} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

where $\quad \bar{V} = \textstyle\sum_c \bar{q}_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$

It is assumed that an estimate exists of a demand function for [illegible]up with a price index as *a* function of a quantity index, as in equations (12) and (13). Assume for a moment that no substitution occurs among commodities (i.*e*., that they are consumed in the fixed observed proportions), and that the demand function is segmented in S segments. Then this case corresponds to Tableau 5, which is a simple extension of the single product case. Only the selling activities are shown.

Table 4: C Commodities in Fixed Proportions

Selling Activities

Objective function	$W_1 \ldots W_s \ldots W_S$	(Max)
Income constraint	$R_1 \ldots R_s \ldots R_S$	$\geq Y^*$
Commodity balances	$-q_{11} \ldots -q_{s1} \ldots -q_{S1}$	≥ 0
	$-q_{12} \ldots -q_{s2} \ldots -q_{S2}$	≥ 0
	$\vdots \quad \vdots \quad \vdots$	
	$-q_{1c} \ldots -q_{sc} \ldots -q_{Sc}$	≥ 0
	$1 \ldots 1 \ldots 1$	≤ 1

In Tableau 6, it is evident that

$$q_{sc} = \bar{a}_c V_s \quad \ldots\ldots\ldots\ldots\ldots (15)$$

where $a_c = \bar{q}_c / \bar{V}_s$, the observed proporiion in physical units of the c^{th} commodity and V_s is the total quantity sold in the s^{th} segment. in physical units. W_s and R_s are, of course, computed from the demand function with appropriate price and quantity indexes, although in Tableau 6, the weights are all constant. The price-weighted total quantity is:

$$\bar{q}_s^* = \sum_c \rho_c q_{sc} = V_s \cdot \Sigma \bar{a}_c \rho_c \ldots\ldots (16)$$

To extend the case of demand in fixed proportions within a group, it is supposed that, for C commodities, the set of feasible alternative mixes, as proportions in physical terms, is given by the matrix A, assumed for simplicity to be invariant across segments.

$$A = \left[a_{cm} \right] \quad \ldots\ldots\ldots\ldots \quad (17)$$

where c = 1, ..., C commodities in the group;

m = 1, ..., M mixes of the commodities; and

a_{cm} is the proportion in physical terms of the c^{th} commodity in the m^{th} mix, such that

$\sum_c a_{cm} = 1$.

The elements, a_{cm}, define the rays shown in Figure 2.

The elements in matrix Q_s in equation set (10) can now be defined as:

$$q_{scm} = a_{cm} V_s \sum_c \bar{a}_c \rho_c / \sum_c a_{cm} \rho_c \quad \ldots\ldots \quad (18)$$

which differs from the expression for q_{sc} (consumption in fixed proportions) in equation (15) by the factor $\sum_c \bar{a}_c \rho_c / \sum_c a_{cm} \rho_c$ which reflects the changing commodity weights. Using equation (16), equation (18) can be re-written as

$$q_{scm} = a_{cm} \bar{q}^*_s / \sum_c a_{cm} \rho_c \quad \ldots\ldots\ldots\ldots (19)$$

The price-weighted total quantity, q^*_{sm}, is given by:

$$q^*_{sm} = \sum_c \rho_c q_{scm}$$
$$= \bar{q}^*_s \quad \ldots\ldots (20)$$

That is, the price weighted quantity of the aggregate commodity is independent of the commodity mix, and it can be written as q^*_s. Using this result, equation (18) can be simplified as follows:

$$q_{scm} = a_{cm} q^*_s / \sum_c a_{cm} \rho_c \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

This completes the definition of the elements of the matrix Qs in equation set (10). By equation (20), q^*_s is invariant with respect to the commodity mix, so that the elements of w_s and r_s are invariant over the mixing activities. They are computed exactly as in the single product case, using, however, q^*_s in place of q_s. To re-capitulate, if the demand function is linear, then:

$$w_s = q^*_s \,(a - 1/2\, bq^*_s) \text{ and}$$
$$r_s = q^*_s \,(a - bq^*_s)$$

The demand side of a planning model may be constructed to incorporate a number of product groups, some of which can consist of a single commodity. Between product groups, the MRS is zero, and within, it is constant, and given by the inverse of the price ratio. It is this last property which leads to the constancy of consumer surplus $(w_s - r_s)$ and of consumer expenditure (r_s) within a commodity group.

The constancy of the MRS can readily be shown for the case of two products, shown in Tableau 7, where again only the selling activities are included.

Tableau 7c Two Comodities in a Segment

Activity level	x_1	x_2	
Objective function	w_s	w_s	(Max)
Income constraint	r_s	r_s	3 Y*
Commodity balances	$-q_{s11}$	$-q_{s12}$	≥ 0
	$-q_{s21}$	$-q_{s22}$	≥ 0
Convex combination constraint	1	1	≤ 1

By the constancy of w_s and r_s, movement along a given indifference function requires changes in the activity levels, x_1 and x_2, which are equal but of opposite sign. Without lack of generality, consider the two cases ($x_1 = 1$, $x_2 = 0$) anti ($x_1 = 0$, $x_2 = 1$) Then the MRS is given by equation (22), dropping the subscript *s*, and q_s^* which is common to all terms.

$$MRS = \frac{\Delta_1}{\Delta_2} = \frac{q_{11} - q_{12}}{q_{21} - q_{22}}$$

$$= \frac{a_{11} / \sum_c a_{c1}\rho_c - a_{12} / \Sigma\, a_{c2}\rho_c}{a_{21} / \Sigma_1\, a_{c1}\rho_c - a_{22} / \Sigma\, a_{c2}\rho_c} \quad \dots\dots\dots\dots\dots \quad (22)$$

By expanding and re-arranging (22),

$$\Delta_1 / \Delta_2 = - \rho_2/\rho_1 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (23)$$

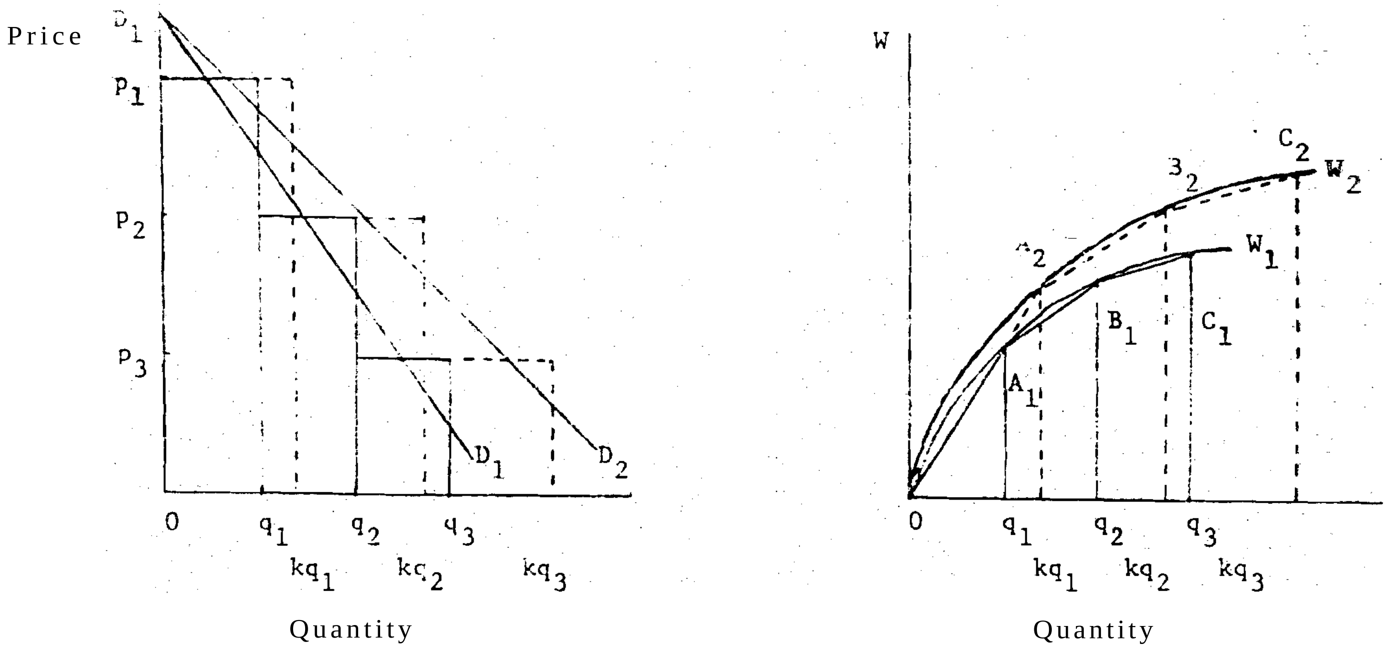
which is the required result,

Comparative Statics

This specification of commodity demand structures incorporates one characteristic which makes it particularly convenient for obtaining comparative statics solutions. This is that the demand function, for any commodity group, can be rotated merely by an appropriate change in the constraint value of the convex combination inequality, i.e., the matrices W_s, R_s, Q_s are invariant under this class of transformations of the commodity demand function.

The transformation of the demand function, for a single product is illustrated in Figure 3, assuming that the function is linear.

Figure 3



The original demand function and corresponding W function is shown as D_1 D_1 and OW_1 respectively, and the rotated demand function and corresponding W function by D_1 D_2 and OW_2 respectively. If the original demand function is

$$p = f(q) \quad \ldots\ldots\ldots\ldots\ldots \quad (24)$$

it is required that the transformed function can be expressed as

$$p = f(kq) \quad \ldots\ldots\ldots\ldots\ldots \quad (25)$$

Such a formulation readily accomodates shifts in the demand function due e.g., to changes in population and/or <u>per capita</u> incomes. The rotation upwards of the demand function is expressed as a proportional lengthening of the segments, with price held constant. For the segmented W function, the slope of the linearized function in each segment, being the approximation to price within that segment, is equal for both W_1 and W_2 for corresponding segments. A similar condition holds for the linearized R function, where the slopes are approximations to marginal revenue within the segment. Given linearity, and the constancy of the slopes of the segmented functions within each segment, the coefficients in the W_s and R_s matrices can bc expressed as simple multiples of the corresponding quantities. This is lone, for the transformed demand function, in Tableau 8, where again only the selling activities appear.

Tableau 8: Transformed Demsnd Function

Objective function	$kq_1w'_1$ ... $kq_sw'_s$ ... $kq_Sw'_S$	(Max)
Income constraint	$kq_1r'_1$ ... $kq_sr'_s$ ... $kq_Sr'_S$	$\geq Y^*$
Commodity balance	$-kq_1$ ... $-kq_s$ ... $-kq_S$	≥ 0
Convex combination constraint	1 ... 1 ... 1	≤ 1

where w'_s and r'_s are simply w_s and r_s divided by q_s; and

k is the factor of proportionality by which the quantity demanded increases at a given price.

Simply by dividing ail the elements of each activity by k, and multiplying through the convex combination constraint by k, the program with the transformed demand function in Tableau 8 can be seen to reduce to a program with coeificients in the constraint matrix identical to those before the demand transformation, but with k replacing unity on the right hand side of the convex combination constraint. This result is readily extended to the commodity group case, as can be seen by replacing q_s with q_c^*, in the objective function and income constraint, by replacing w'_s and r'_s [illegible] the corresponding vectors W'_s and R'_s 1/, and by recalling that the matrices [illegible] in the commodity balances can be written as scalar multiples of q_c^*. This characteristic of the demand structure permits computationally simple parametric variation of the position of the demand function. It also opens the possibility, in a larger system, of endogenously determining both the position of the demand functions and tne position on it.

1/ q_c^*, being [illegible] over mixing activities, i s a scalar.

Intercational Trade

A representation of international trade can readily be incorporated into the structures developed in this article in the usual way in which it is incorporated into planning models, that is, by adding commodity-specific importing activities as additional "production" activities and similarly by adding exporting activities as additional selling activities. Again, as usual, it is possible to specify import supply (export demand) as being infinitely elastic, as being infinitely elastic but bounded, or as being represented by an upward sloping supply (downward sloping demand) schedule. In this last case, it is possible to approximate the non-linearities involved by the methods developed above. Notice, however, that it is only possible to specify a monopolistic formulation of export supply, or a monopsonistic formulation of import demand, unless the objective function and the scope of the model represents multi-country welfare.

When trading opportunities are included as outlined above, the model captures the different trading positions posited by price theory and depending on relative domestic and foreign supply and demand functions and on whether the objective function is chosen to ,reflect competitive or mono- , polistic behaviour. For cxamplc, in the monopolist case, final product ,importing activities never enter the optimal basis, and the model reproduces the expected two-price behaviour when the foreign marginal revenue function lies above the domestic marginal revenue function. 1/

1/ One case which the structure will not handle is the monopolist case where either of the demand functions is of the double-log form and where the elasticity of demand is less than unity in absolute value. In this case, marginal revenue is negative, but increasing, i.e. the function is non-convex.

An Application

Some numerical results with these demand structures were obtained with the model for Mexican agriculture, CHAC. 1/ Here it is relevant to note the trade-offs between sector income and consumers' "welfare" which occurs in moving from the competitive to the monopolistic solution. In CHAC, two definitions of gains to producers were used: farmers' income and farmers' profits. The former includes own-wages, whereas the latter does not. Proper representation of monopolistic behaviour requires maximization of producers' profits rather than total income.

Successively higher lower bounds were place on farmers' profits to simulate supply control regimes, and finally, at the extreme, farmers' profits were maximized. The following table shows the numerical results:

Table 1.

Percentage change in: a/	Supply Control Solutions I	II	III	Monopolistic solution
Objective function	-0.1	-0.1	-0.1	-0.2
Farmers' profits	10.0 b/	30.0 b/	50.0 b/	93.0
Farmers' income	6.3	19.3	32.2	57.0
Consumer surplus	-1.5	-4.3	-7.3	-13.1
Production	-0.3	-0.6	-0.9	-1.3

a/ Relative to competitive solution.

b/ Binding constraint on farmers' profits.

From the table, it may be seen that there is very little "welfare loss" as measured crudely by aggregating the sun of consumer and producer

1/ See Duloy and Norton (1972 (a) and (b)).

surpluses, in moving froa the competitive to the monopolistic solution.[1/] There is , however, a marked increase in farmers' income and a corresponding decrease in consumer surplus.

Conclusions

First, this paper has developed a practical procedure for enforcing both competitive and noncompetitive market st·uctures by means of the optimization inherent in linear programming. The procedure has the property that arbitrarily close approximations to nonlinear forms - in both the objective function and constraint set - can be made without much loss of the computational efficiency of the simplex algorithm. In this respect, the paper extends the ideas of Miller (1963) and integrates them with the work on market forms in mathematical programming by Samuelson (1952) and Takayama and Judge (1964, 1971).

Second, it has shown that the noncompetitive market structure nay be used for measuring income at endogenous prices in a competitive model, and thus may serve as a constraint on that measure of income to represent certain classes of economic policies. Third, it has developed a procedure for approximating product substitution effects in demand in a linear program. Alternative procedures are presented for the cases of full and pertial information on the matrix of own- and cross-price elasticities. The demand structure can very simply be transformed to take account of any shift in demand which can be represented by a rotation of the demand function.

Fourth, international trade can be integrated into the structure.

1/ Plessner (1971) finds that rather more significant welfares losses would be implied by monopoly in the apple and pear markets in Israel.

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